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REMARKS ON FIXED POINT THEOREMS ON STAR-SHAPED SETS

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A large number of fixed point theorems without convexity conditions are available in analysis. Some of them are related to the concept of star-shapedness.

In this paper, we obtain a fixed point theorem which extends known results of Meir-Keeler [13], Boyd-Wong [5], Browder [6], and the Banach contraction principle. Using this, we extend and unify fixed point theorems of Assad [3], Assad-Kirk [1], Kuhfittig [11], and Dotson [8] on convex or star-shaped sets. Finally, related results of Rhoades [14] and Assad [2] on convex sets are also substantially extended to star-shaped sets.

Let X be a complete metric space and $K\subset X$. We say that a map $T: K\to X$ is metrically inward if for each $x\in K$ there exists an element u of K such that $d(x,u)+d(u,Tx)=d(x,Tx)$ where $u=x$ iff $x=Tx$ [7].

Let $x_0\in K$. We shall construct two sequences $\{x_n\}$, $\{x_n'\}$ as follows: Define $x_1'=Tx_0$. If $x_1'\in K$, set $x_1=x_1'$. If $x_1'\notin K$, choose $x_1\in K$ so that $d(x_0,x_1)+d(x_1,x_1')=d(x_0,x_1')$. Set $x_2'=Tx_1$. If $x_2'\in K$, set $x_2=x_2'$. If not, choose $x_2\in K$ so that $d(x_1,x_2)+d(x_2,x_2')=d(x_1,x_2')$. Continuing in this manner, we obtain $\{x_n\}$, $\{x_n'\}$ satisfying

(1) $x_{n+1}'=Tx_n$,

(2) $x_n=x_n'$ if $x_n'\in K$, and

(3) $x_n\in K$ and $d(x_{n-1},x_n)+d(x_n,x_n')=d(x_{n-1},x_n')$ if $x_n'\notin K$.

Let $P=\{x_i\in\{x_n\}\mid x_i=x_i'\}$ and $Q=\{x_i\in\{x_n\}\mid x_i\neq x_i'\}$.

The following is our main result which is comparable to Theorem 2.2 of Caristi [7].

THEOREM 1. *Let (X,d) be a complete metric space, K a nonempty closed subset of X, and $T: K\to X$ a metrically inward map satisfying the condition:*

(A) *given $\varepsilon>0$, there exists a $\delta>0$ such that $x,y\in K$ and*

$$\varepsilon\leqslant d(x,y)<\varepsilon+\delta \text{ imply } d(Tx,Ty)<\varepsilon.$$

If a sequence $\{x_n\}$ defined as above satisfies the condition:

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(*) *$x_n \in Q$ implies $x_{n-1}, x_{n+1} \in P$,*

then T has a unique fixed point and $\{x_n\}$ converges to the fixed point.

Proof. If there exists a j such that $x_n \in P$ for all $n \geqslant j$, then $\{x_n\}$ converges to a fixed point of T by the method of Meir-Keeler [13]. Therefore, we may assume that Q contains infinitely many elements. Let $Q = \{x_{n_k}\}$. If $x_j = x_{j+1}$ for some j, then $x_j = Tx_j$ is a fixed point. Therefore, we may assume $x_n \neq x_{n+1}$ and hence $x_n \neq x_{n+1}'$ for all n.

Step 1. $d(x_{n_k-1}, x_{n_k}') \to 0$ as $k \to \infty$.

Set $n_k = r$ and $n_{k+1} = s$. Since T is contractive and $r < s-1$, we have

$$\begin{aligned} d(x_{s-1}, x_s') &= d(Tx_{s-2}, Tx_{s-1}) < d(x_{s-2}, x_{s-1}) < \cdots \\ &< d(x_r, x_{r+1}) \leqslant d(x_r, x_r') + d(x_r', x_{r+1}) \\ &= d(x_r, x_r') + d(Tx_{r-1}, Tx_r) \\ &< d(x_r, x_r') + d(x_{r-1}, x_r) \\ &= d(x_{r-1}, x_r'). \end{aligned}$$

Therefore $\{d(x_{r-1}, x_r')\}$ is decreasing to a limit $\varepsilon \geqslant 0$. Suppose $\varepsilon > 0$. Then there exists a $\delta > 0$ satisfying (A), and hence there exists an N such that $\varepsilon \leqslant d(x_{n_k-1}, x_{n_k}') < \varepsilon + \delta$ for all $k \geqslant N$. Since $d(x_{s-1}, x_s') < d(x_r, x_{r+1}) < d(x_{r-1}, x_r')$, we have $\varepsilon \leqslant d(x_r, x_{r+1}) < \varepsilon + \delta$, and hence $d(Tx_r, Tx_{r+1}) < \varepsilon$ by (A). On the other hand,

$$\begin{aligned} d(x_{s-1}, x_s') &= d(Tx_{s-2}, Tx_{s-1}) < d(x_{s-2}, x_{s-1}) = d(x_{s-2}, x_{s-1}') \\ &< \ldots < d(x_{r+1}, x_{r+2}') = d(Tx_r, Tx_{r+1}) < \varepsilon, \end{aligned}$$

which is a contradiction.

Step 2. $d(x_n, x_{n+1}) \to 0$, $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Suppose $d(x_n, x_{n+1}) \to 0$ is false. Then there exists an $\varepsilon > 0$ such that for every $N \geqslant 0$, there exists an $n \geqslant N$ such that $d(x_{n-1}, x_n) > \varepsilon$. By Step 1, there exists an $M \geqslant 0$ such that $d(x_{n_k-1}, x_{n_k}') < \varepsilon$ for $k \geqslant M$. Let $N = n_k$ for some $k \geqslant M$. Then for all $n \geqslant N$ such that $x_n \in P$, there exists a unique j such that $n_j < n < n_{j+1}$, and hence

$$\begin{aligned} d(x_{n-1}, x_n) &= d(x_{n-1}, x_n') = d(Tx_{n-2}, Tx_{n-1}) < d(x_{n-2}, x_{n-1}) \\ &= d(x_{n-2}, x_{n-1}') < \ldots < d(x_{n_j-1}, x_{n_j}') < \varepsilon, \end{aligned}$$

a contradiction. Similar argument shows that $d(x_n, Tx_n) \to 0$.

Step 3. $\{x_n\}$ is Cauchy.

Suppose not. Then there exists an $\varepsilon>0$ such that $\lim\sup_{m,n} d(x_m, x_n)> 2\varepsilon$. Now there exists a δ, $0<\delta\leqslant\varepsilon$, satisfying (A). By Step 2, there exists an M such that for $n\geqslant M$,

$$d(x_n, x_{n+1})<\delta/3 \text{ and } d(x_n, Tx_n)<\delta/3.$$

Choose $m>n>M$ so that $d(x_m, x_n)>2\varepsilon$, For j, $m\geqslant j\geqslant n$,

$$d(x_m, x_j)\leqslant d(x_m, x_{j+1})+d(x_j, x_{j+1})$$

implies

$$|d(x_m, x_j)-d(x_m, x_{j+1})|\leqslant d(x_j, x_{j+1})<\delta/3.$$

Since $d(x_m, x_{m+1})<\delta/3<\varepsilon$ and $d(x_m, x_n)>2\varepsilon\geqslant\varepsilon+\delta$, there exists a j, $m\geqslant j \geqslant n$, such that $\varepsilon+2\delta/3<d(x_m, x_j)<\varepsilon+\delta$. On the other hand,

$$\begin{aligned} d(x_m, x_j) &\leqslant d(x_m, Tx_m)+d(Tx_m, Tx_j)+d(Tx_j, x_j)\\ &<\delta/3+\varepsilon+\delta/3=\varepsilon+2\delta/3, \end{aligned}$$

a contradiction.

Since $\{x_n\}$ is Cauchy, it converges to some $p\in K$. Since $d(x_n, Tx_n)\to 0$, we have $Tx_n\to p$ and hence $p=Tp$ by the continuity of T. The uniqueness is clear. This completes our proof.

REMARK 1.1. The condition (A) is due to Meir-Keeler [13]. Since every selfmap is metrically inward and satisfies (*), Theorem 1 extends Meir-Keeler's theorem [13]. Note that the condition (A) includes the following:

(A_1) given $\varepsilon>0$, there exists a $\delta>0$ and an ε_0 with $0<\varepsilon_0<\varepsilon$ such that $\varepsilon\leqslant d(x,y)<\varepsilon+\delta$ implies $d(Tx, Ty)\leqslant\varepsilon_0$;

(A_2) given $\varepsilon>0$, there exists a $\delta>0$ and an ε_0 with $0<\varepsilon_0<\varepsilon$ such that $d(x,y)<\varepsilon+\delta$ implies $d(Tx, Ty)\leqslant\varepsilon_0$;

(A_3) T is a Banach contraction, that is, there exists a $k<1$ such that $d(Tx, Ty)\leqslant k\, d(x,y)$.

Note that $(A_3)\Rightarrow(A_2)\Rightarrow(A_1)\Rightarrow(A)$. Hegedüs-Szilágyi [9] noted that (A_1) is equivalent to the contractive type condition of Boyd-Wong [5], and (A_2) to the condition of Browder [6].

REMARK 1.2. Caristi ([4], Theorem 2.2) obtained Theorem 1 for the condition (A_3) instead of (A) without assuming the condition (*). He showed merely the existence of fixed point using the Caristi-Kirk fixed point theorem.

The star-shapedness of linear spaces [11] can be extended for metric spaces as follows:

A metric space (X,d) is said to be star-shaped if there exists at least one point $c\in X$ (called a star-center) such that for each $x\in X$, c and x can be joined by a metric segment of X, that is, a subset isometric to an interval of length $d(c, x)$. A metric space X is said to be star-shaped with respect to a subset K of X if each $y \in K$ is a star-center of X

REMARK 1.3. A metric space X is convex in the sense of Bing [4] if for every $x, y \in X$ there exists a $z \in X$ such that $d(x, y)/2 = d(x, z) = d(z, y)$, and convex in the sense of Menger if for every $x, y \in X$ there exists a $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$. Clearly, the former implies the latter. However, for complete metric spaces those concepts are identical since a theorem of Menger (see [10]) states that a convex complete metric space contains together with x, y also a metric segment whose extremities are x and y.

REMARK 1.4. Matkowski and Wegrzyk [12] claimed that on a complete, metrically convex, metric space, the conditions (A), (A_1), (A_2) and some others are equivalent.

In certain cases, while constructing the sequences $\{x_n\}$, $\{x_n'\}$ as in Theorem 1, if $x_n' \notin K$, x_n can be chosen at the boundary of K. From now on, ∂ denotes the boundary and ∂_X the relative boundary.

THEOREM 2. *Let* (X, d) *be a complete metric space and* K *a closed subset of* X *such that* X *is star-shaped with respect to* K. *If a map* $T : K \to X$ *satisfies the condition* (A) *and* $T(\partial K) \subset K$, *then* T *has a unique fixed point and a sequence* $\{x_n\}$ *defined as above converges to the fixed point.*

Proof. Since X is star-shaped with respect to K, T is metrically inward. Note also that the assumption $T(\partial K) \subset K$ implies (*). Therefore Theorem 2 follows from Theorem 1.

COROLLARY 2.1. (Assad [3]) *Let* (X, d) *be a complete, metrically convex, metric space and* K *a nonempty closed subset of* X. *Suppose that* $T : K \to X$ *satisfies* (A) *and* $T(\partial K) \subset K$. *Then* T *has a unique fixed point.*

COROLLARY 2.2. *Let* E *be a Banach space,* X *a closed subset of* E *and,* K *a closed subset of* X *such that* X *is star-shaped with respect to* K. *If* $T : K \to X$ *satisfies the condition* (A) *and* $T(\partial_X K) \subset K$, *then* T *has a unique fixed point.*

REMARK 2.1. Kuhfittig [11] obtained Corollary 2.2 for the condition (A_3) instead of (A), and Assad-Kirk [1] for the condition (A_3) and convex X.

Following Dotson's method in [8], from Corollary 2.2 we obtain

COROLLARY 2.3. (Kuhfittig [11]) *Let* E *be a Banach space,* X *a closed subset of* E, *and* K *a compact star-shaped subset of* X *such that* X *is star-shaped with respect to* K. *If* $T : K \to X$ *is nonexpansive, and if* $T(\partial_X K) \subset K$, *then* T *has a fixed point.*

REMARK 2.2. Corollary 2.3 extends Theorem 1 of Dotson [8]. Note that other main results in [11] follows from Corollary 2.2.

The following is closely related to Theorem 1 and its proof can be given by modifying that of Rhoades' theorem [14].

THEOREM 3. *Let* (X, d) *be a complete metric space,* K *a closed subset of* X, *and* $T: K \to X$ *a metrically inward map satisfying the condition*:

(B) *there exists an* $h \in (0, 1)$ *such that for each* $x, y \in K$,
$d(Tx, Ty) \leqslant h \max \{d(x, y)/2,\ d(x, Tx),\ d(y, Ty),\ [d(x, Ty) + d(y, Tx)]/q\}$ *where* $q \in [1+2h, \infty)$. *If a sequence* $\{x_n\}$ *defined as in* Theorem 1 *satisfies the condition* (*), *then* T *has a unique fixed point and* $\{x_n\}$ *converges to the point.*

COROLLARY 3.1. *Let* (X, d) *be a complete metric space and* K *a closed subset of* X *such that* X *is star-shaped with respect to* K. *If a map* $T: K \to X$ *satisfies the condition* (B) *and* $T(\partial K) \subset K$, *then* T *has a unique fixed point.*

COROLLARY 3.2. *Let* E *be a Banach space,* X *a closed subset of* E, *and* K *a closed subset of* X *such that* X *is star-shaped with respectto* K. *If* $T: K \to X$ *satisfies the condition* (B) *and* $T(\partial_X K) \subset K$, *then* T *has a unique fixed point in* K.

REMARK 3. Rhoades [14] obtained Corollary 3.2 for convex X, and Assad [2] for T satisfying the condition

$$(B_1) \qquad \|Tx - Ty\| \leqslant k(\|x - Tx\| + \|y - Ty\|),\ k < 1/2.$$

Note that (B_1) implies (B).

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