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GAUSS MAPS ON SUBMANIFOLDS OF RIEMANNIAN SPACES

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§ 0. Introduction

For a hypersurface N in the euclidean space E^m, we have the well known Gauss map from N to the unit hypersphere S and the ratio of volume elements at corresponding points of S and N is (eventually up to the sign) equal to the Gaussian curvature of N.

If N is a hypersurface of a general Riemannian manifold R^m, then we can work in the following way: suppose that η is a unit normal vector field on N with domain U. If $\tilde{D}$ is the Riemannian connection of R^m and if X is a vector field of N then the Weingarten map L is given by

$$\tilde{D}_X\eta = L(X).$$

L is a self-adjoint linear map $N_p \to N_p$ at each point p. Assume that det $L=$ Gaussian curvature G of $N \neq 0$ at each point of U. The metric tensor on R^m and the induced metric tensor on N are denoted by g. Define on the differentiable manifold U:

$$\bar{g}(X_p, Y_p) = g(L(X_p), L(Y_p))$$

for all vectors X_p, Y_p at each point p of U.

In this way, we have a new Riemannian manifold $\bar{U}$ with metric tensor $\bar{g}$ and we can locally define the Gauss map on N in R^m as the natural bijection $i : U \to \bar{U}$; $p \to p$. If i^* is the dual linear map of the Jacobian i_* of i and if ω (resp. $\bar{\omega}$) is a volume element of N (resp. $\bar{U}$) at p, it is easy to see that $i^*(\bar{\omega}) = |G(p)|\omega$. Suppose now that N is an n-dimensional submanifold of an m-dimensional Riemannian manifold R^m and that η is a normal vector field on N with domain U, which is parallel in the normal bundle $N^\perp$, that is, $D_X{}^\perp\eta = 0$ for all vector fields X of N. Then we have, if $\tilde{D}$ is the Riemannian connection of R^m:

$$\tilde{D}_X\eta = -A_\eta(X),$$

where A_η is a self-adjoint linear map at each point. Moreover, assume that det $A_{\eta_p} \neq 0$ at each point p of U and define on the differentiable manifold U (g is again the induced metric tensor on N):

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$\bar{g}(X_p, Y_p) = g(A_{\eta_p}(X_p), A_{\eta_p}(Y_p))$

for all vectors X_p, Y_p at each point p of U.

We have again a new Riemannian manifold $\bar{U}$ with metric tensor $\bar{g}$ and we can define the Gauss map with respect to η in the same way as before. Moreover, it is not difficult to see that now $i^*(\bar{\omega}) = |\det A_{\eta_p}|\, \omega$ at each point p of U and $\det A_{\eta_p}$ is the Gauss curvature of N relative to η_p or the Lipschitz–Killing curvature of N in the direction of η_p.

The main purpose of the present paper is to realize in an analogous way Gauss maps on submanifolds of Riemannian manifolds, but with respect to non-parallel normal sections (especially non-parallel geodesic normal sections). In §1 we obtain a geometrical interpretation of the connection between the volume elements $\bar{\omega}$ and ω. In §2 we treat a special case where the "spherical image" $\bar{U}$ has constant curvature $+1$. Finally, we give some examples in §3.

§1. The Gauss map with respect to a non-parallel geodesic normal section η

We shall assume throughout that all manifolds, maps, vector fields, etc... are differentiable of class C^∞. Suppose that N is an n-dimensional submanifold of a Riemannian manifold R^m $(n \geqslant 2; m > n)$. A unit normal vector field η on N is called a normal section on N. If $\tilde{D}$ (resp. D) is the Riemannian connection of R^m (resp. N) and if X is a vector field of N, the Weingarten equation says

$$\tilde{D}_X \eta = -A_\eta(X) + D_X{}^\perp \eta.$$

η is a geodesic section on N if A_η is everywhere zero. In this case N is said to be geodesic with respect to η. We call η a non-parallel normal section on N, if $D^\perp \eta$ is nowhere zero, that is, if $D_{X_p}{}^\perp \eta \neq 0$ for each vector $X_p \neq 0$ at each point p of the domain U of η. Thus, we have for a geodesic non-parallel normal section η on N:

$$\tilde{D}_X{}^\perp \eta = D_X{}^\perp \eta \tag{1.1}$$

and this is never zero if X is nowhere zero. Remark that for each point p of the domain U of η and for each vector X_p of N_p, $D_{X_p}{}^\perp \eta$ is orthogonal with N_p and with η_p. Moreover $D_{X_p}{}^\perp \eta$ is linear in X_p, so if η is non-parallel in the normal bundle $N^\perp$, the codimension of N is at least $n+1$, that is, $m \geqslant 2n+1$. For the metric tensor of R^m and the induced metric tensor on N, we use the same notation g.

DEFINITION. Suppose that η is a geodesic non-parallel normal section on N

with domain U. For each point p of U and all vectors X_p and Y_p of N_p, we set

$$\bar{g}(X_p, Y_p)=g(\tilde{D}_{X_p}\eta, \tilde{D}_{Y_p}\eta). \tag{1.2}$$

Since η is non-parallel, $\bar{g}$ is a metric tensor on the differentiable manifold U, which becomes a new Riemannian manifold $\bar{U}$. The Gauss map on N with respect to η is locally given by the natural bijection $i: U\rightarrow\bar{U}$; $p\rightarrow p$.

Next, consider an $(n+1)$-dimensional submanifold C of R^m, a point p of C and a fixed unit vector ξ_p of C_p. If V is the (vector valued) second fundamental form of C in R^m, then the Riemann curvatures $K(\xi_p, X_p)$ and $\tilde{K}(\xi_p, X_p)$ of respectively C and R^m, in the two-dimensional direction spanned by ξ_p and an orthogonal unit vector X_p of C_p, are connected by:

$$\begin{aligned}K(\xi_p, X_p)=\tilde{K}(\xi_p, X_p)&+g(V(\xi_p, \xi_p), V(X_p, X_p))\\&-g(V(\xi_p, X_p), V(\xi_p, X_p)).\end{aligned} \tag{1.3}$$

Remark that $g(V(\xi_p, \xi_p), V(X_p, Y_p))-g(V(\xi_p, X_p), V(\xi_p, Y_p))$ (with also $Y_p \perp \xi_p$ in C_p) determines a two-covariant symmetric tensor on the orthogonal complement of ξ_p in C_p. So, we can find in C_p n mutually orthogonal unit vectors $e_p{}^1, \dots, e_p{}^n$ which are orthogonal with ξ_p such that $K^i(\xi_p)=K(\xi_p, e_p{}^i)-\tilde{K}(\xi_p, e_p{}^i)$ $i=1, \dots, n$ are extremal values for this difference of curvature.

DEFINITION. We say that each $e_p{}^i$ determines a principal direction of C_p with respect to ξ_p and we set

$$K(\xi_p)=\prod_{i=1}^{n} K^i(\xi_p).$$

Now we return to our n-dimensional submanifold N with his geodesic non-parallel section η with domain U. Through each point p of U we have locally a unique geodesic of R^m which has η_p as tangent vector at p. Assume that these (parts of) geodesics describe an $(n+1)$-dimensional submanifold C of R^m which contains $U\subset N$ as a hypersurface (n-dimensional submanifold). If R^m is the euclidean m-space and $n=2$, then C becomes a (part of a) congruence of straight lines.

We indentify from now on vectors and forms of U and $\bar{U}$ so we do not use the Jacobian i_* and the dual linear map i^*.

THEOREM 1.1. 1° *At each point p of U the principal directions of C_p with respect to η_p are orthogonal in both Riemannian manifolds U and $\bar{U}$.*

2° *Suppose that $\sigma: [a, b]\rightarrow U$ is a curve on N with arc length s in U and $\bar{s}$ in $\bar{U}$, and with unit (in U) tangent vector T_p at p, then*

$$\left(\frac{d\bar{s}}{ds}\right)_p^2=\tilde{K}(\eta_p, T_p)-K(\eta_p, T_p).$$

3° *If* $\bar{\omega}$ (resp. ω) *is a volume-element of* $\bar{U}$ (resp. U) *at a point* p *of* U, *then*

$$\bar{\omega}=\sqrt{(-1)^n K(\eta_p)}\,\omega.$$

Proof: For a geodesic line of R^m, with tangent vector field P, which lies on C, we have, if D is the Riemannian connection of C and if V is the second fundamental tensor of C in R^m:

$$\tilde{D}_P P=D_P P+V(P,P)=0,$$

so $V(P,P)=0$, i. e. it is an asymptotic line of C. In particular we will have

$$V(\eta_p,\eta_p)=0, \text{ at each point } p\in U. \tag{1.4}$$

Suppose that X is a vector field of U, then we have the Gauss equation

$$\tilde{D}_X\eta=D_X\eta+V(X,\eta),$$

but we know from (1. 1) that

$$\tilde{D}_X\eta=D_X{}^{\perp}\eta,$$

so we find $D_X\eta=0$ for each vector field X of U, which means that U is totally geodesic in C. On the other hand we have

$$\tilde{D}_X\eta=V(X,\eta).$$

Because of this, the first part of the theorem follows immediately from (1. 2), (1. 3) and (1. 4).

From all this, we also know that if e_p is a unit (in U) vector of N_p, then

$$\bar{g}(e_p,e_p)=\tilde{K}(\eta_p,e_p)-K(\eta_p,e_p). \tag{1.5}$$

Since the arc length $\bar{s}$ in $\bar{U}$ of the curve σ with tangent vector T is given by

$$\bar{s}=\int\sqrt{\bar{g}(T,T)}\,ds.$$

The second part of the theorem follows from (1. 5).

Finally consider a base $e_p{}^1,\dots,e_p{}^n$ of N_p $(p\in U)$ which is orthogonal in $\bar{U}$ and orthonormal in U. The dual forms are denoted by $\omega_1,\dots,\omega_n$. The vectors $e_p{}^i/\sqrt{\bar{g}(e_p{}^i,e_p{}^i)}$ form an orthonormal base of $\bar{U}_p$ with corresponding dual forms $\bar{\omega}_1,\dots,\bar{\omega}_n$. It is easy to see that $\bar{\omega}_i=\sqrt{\bar{g}(e_p{}^i,e_p{}^i)}\,\omega_i$ and we get

$$\bar{\omega}=\bar{\omega}_1\wedge\cdots\wedge\bar{\omega}_n=\sqrt{\prod_{i=1}^{n}\bar{g}(e_p{}^i,e_p{}^i)}\,\omega_1\wedge\cdots\wedge\omega_n=\sqrt{(-1)^n K(\eta_p)}\,\omega.$$

THEOREM 1.2. *The Gauss map* $i : U \to \bar{U}$ *is conformal* (resp. *homothetic*, i.e. *isometric up to a constant*) *iff* $\tilde{K}(\eta_p, e_p) - K(\eta_p, e_p)$ *is independent of* $e_p(\neq 0) \in U_p$ *at each point* p *of* U (resp. *iff this expression is independent of* e_p $(\neq 0) \in U_p$ *and of* $p \in U$).

Proof: i is conformal iff the metrics g and $\bar{g}$ are proportional, that is, iff there is a positive function $\gamma : U \to R$ such that $\bar{g} = \gamma g$. If e_p is a U-unit vector in U_p, we get from (1.5) $\gamma(p) = \tilde{K}(\eta_p, e_p) - K(\eta_p, e_p)$. Moreover, i is homothetic iff γ is a (positive) constant function on U. This completes the proof.

A normal section δ on N is said to be umbilical if A_δ is everywhere proportional to the identity transformation I, i.e. $A_\delta = \alpha I$ for some function α. Assume that $\tilde{D}_{X_p}\delta \neq 0$ for each vector $X_p \neq 0$ of U_p at each point p of U and define for each point p of the domain U of δ and for all vectors X_p and Y_p of N_p:

$$\bar{\bar{g}}(X_p, Y_p) = g(\tilde{D}_{X_p}\delta, \tilde{D}_{Y_p}\delta).$$

$\bar{\bar{g}}$ is a metric tensor on the differentiable manifold U, which becomes a new Riemannian manifold $\bar{\bar{U}}$.

We can work in an analogous way as before, but instead of (1.5), we find now, if e_p is a unit vector (in U) of N_p:

$$\begin{aligned} \bar{\bar{g}}(e_p, e_p) &= g(A_{\delta_p}(e_p), A_{\delta_p}(e_p)) + g(D_{e_p}{}^{\perp}\delta, D_{e_p}{}^{\perp}\delta) \\ &= g(\alpha(p)e_p, \alpha(p)e_p) + g(V(\delta_p, e_p), V(\delta_p, e_p)) \\ &= \alpha^2(p) + \tilde{K}(\delta_p, e_p) - K(\delta_p, e_p). \end{aligned} \tag{1.6}$$

Remark that if $n=2$, $\alpha^2(p) = \det A_{\delta_p}$ is the Lipschitz-Killing curvature of N at p in the direction of δ_p.

More generally, we have for two vectors X_p and Y_p of U_p:

$$\bar{\bar{g}}(X_p, Y_p) = \alpha^2(p) g(X_p, Y_p) + g(V(\delta_p, X_p), V(\delta_p, Y_p)).$$

So, for such umbilical normal section, the first part of theorem 1.1. remains true if we write δ_p and $\bar{\bar{U}}$ instead of η_p and $\bar{U}$. For the second part we find now ($\bar{\bar{s}}$ is the arc length of the curve in $\bar{\bar{U}}$):

$$\left(\frac{d\bar{\bar{s}}}{ds}\right)_p^2 = \alpha^2(p) + \tilde{K}(\delta_p, T_p) - K(\delta_p, T_p).$$

The connection between volume-elements $\bar{\bar{\omega}}$ of $\bar{\bar{U}}$ and ω of U at a point p of U is given by

$$\bar{\bar{\omega}} = \sqrt{\prod_{i=1}^{n} (\alpha^2(p) - K^i(\delta_p))}\,\omega.$$

From (1.6) we see that the condition for conformality of the new Gauss map $i : U \to \bar{\bar{U}}$ remains the same as before, but this Gauss map will be homothetic iff $\alpha^2(p) + \tilde{K}(\delta_p, e_p) - K(\delta_p, e_p)$ is independent of the unit (in U) vector e_p of U_p and independent of $p \in U$.

Suppose now that ξ is any non-parallel normal section on N with domain U. We can define

$$\bar{\bar{g}}(X_p, Y_p) = g(D_{X_p}{}^{\perp}\xi, D_{Y_p}{}^{\perp}\xi). \tag{1.7}$$

We find a new Riemannian manifold $\bar{\bar{U}}$ and the corresponding map : $U \to \bar{\bar{U}}; p \to p$. Moreover, theorem 1.1. and 1.2. remain true for this manifold $\bar{\bar{U}}$. If we work in the euclidean space E^m, then, instead of working with $\bar{U}$ and $\bar{\bar{U}}$, we can use the spherical image of N (or of U) with respect to η and δ which is the n-dimensional submanifold (of a unit $(m-1)$-sphere) determined by the endpoints of η (resp. δ) which we bind at a fixed point of space. But if we work with the metric $\bar{\bar{g}}$ defined by (1.7), the manifold $\bar{\bar{U}}$ is in general no more (locally) isometric with the spherical image of U with respect to ξ.

REMARKS. 1. If N is totally geodesic (resp. totally umbilical) in R^m, then of course every normal section on N is geodesic (resp. umbilical).

2. If we construct the manifold C for a parallel normal section θ with domain U on N, then we have $\tilde{K}(\theta_p, e_p) = K(\theta_p, e_p)$ for each vector e_p of U_p at each point p of U.

3. It is easy to see that in the "umbilical case", U is an umbilical hypersurface in C.

§2. *N* has flat normal connection in R^m

In this section we suppose that we have again a geodesic non-parallel normal section η with domain U on N and that N has flat normal connection in R^m. $\bar{g}$ is again defined by (1.2). Assume that at each point p of U, $e_p{}^1, \ldots, e_p{}^n$ is a base of U_p which is orthonormal for g and orthogonal for $\bar{g}$. The $(n+1)$-dimensional subbundle of $U^{\perp}$ at each point p of U spanned by η_p and $De_p^i{}^{\perp}\eta$ $i=1, \cdots, n$ is denoted by $U_1{}^{\perp}$. Recall that such subbundle is parallel in the normal bundle $U^{\perp}$ if for each vector field φ of $U_1{}^{\perp}$ and for each vector field X of U, $D_X{}^{\perp}\varphi$ is a vector field of $U_1{}^{\perp}$.

The following theorems are also correct for a general nonparallel normal section ξ on N if we use the definition $\bar{\bar{g}}(X_p, Y_p) = g(D_{X_p}{}^{\perp}\xi, D_{Y_p}{}^{\perp}\xi)$ for the new metric tensor in U.

THEOREM 2. 1. *Suppose that the subbundle $U_1{}^{\perp}$ is parallel in the normal bundle $U^{\perp}$. Then $\bar{U}$ is a space of constant curvature* $+1$.

Proof. The Riemannian connection $\bar{D}$ of the Riemannian manifold $\bar{U}$ is determined by (X, Y, Z are any vector fields of U):

$$2\bar{g}(\bar{D}_X Y, Z) = X\bar{g}(Y, Z) + Y\bar{g}(X, Z) - Z\bar{g}(X, Y) + \bar{g}([Z, X], Y) + g([Z, Y], X) + \bar{g}([X, Y], Z). \quad (2.1)$$

We know that since η is geodesic, $\bar{g}(X, Y) = g(D_X{}^{\perp}\eta, D_Y{}^{\perp}\eta)$ and moreover, since N has flat normal connection in R^m, we have for each two vector fields X and Y of U:

$$D_X{}^{\perp}D_Y{}^{\perp}\eta - D_Y{}^{\perp}D_X{}^{\perp}\eta = D_{(X,Y)}{}^{\perp}\eta.$$

So, we find from (2.1), using also the property

$$Xg(D_Y{}^{\perp}\eta, D_Z{}^{\perp}\eta) = g(D_X{}^{\perp}D_Y{}^{\perp}\eta, D_Z{}^{\perp}\eta) + g(D_Y{}^{\perp}\eta, D_X{}^{\perp}D_Z{}^{\perp}\eta),$$

$$\begin{aligned} 2\bar{g}(\bar{D}_X Y, Z) &= g(D_X{}^{\perp}D_Y{}^{\perp}\eta, D_Z{}^{\perp}\eta) + g(D_Y{}^{\perp}\eta, D_X{}^{\perp}D_Z{}^{\perp}\eta) + g(D_Y{}^{\perp}D_X{}^{\perp}\eta, D_Z{}^{\perp}\eta) \\ &+ g(D_X{}^{\perp}\eta, D_Y{}^{\perp}D_Z{}^{\perp}\eta) - g(D_Z{}^{\perp}D_X{}^{\perp}\eta, D_Y{}^{\perp}\eta) - g(D_X{}^{\perp}\eta, D_Z{}^{\perp}D_Y{}^{\perp}\eta) \\ &+ g(D_Z{}^{\perp}D_X{}^{\perp}\eta, D_Y{}^{\perp}\eta) - g(D_X{}^{\perp}D_Z{}^{\perp}\eta, D_Y{}^{\perp}\eta) + g(D_Z{}^{\perp}D_Y{}^{\perp}\eta, D_X{}^{\perp}\eta) \\ &- g(D_Y{}^{\perp}D_Z{}^{\perp}\eta, D_X{}^{\perp}\eta) + g(D_X{}^{\perp}D_Y{}^{\perp}\eta, D_Z{}^{\perp}\eta) - g(D_Y{}^{\perp}D_X{}^{\perp}\eta, D_Z{}^{\perp}\eta) \\ &= 2g(D_X{}^{\perp}D_Y{}^{\perp}\eta, D_Z{}^{\perp}\eta). \end{aligned} \quad (2.2)$$

Suppose that $\bar{e}_1, \ldots, \bar{e}_n$ is an orthonormal base field of $\bar{U}$ (thus $\bar{g}(\bar{e}_i, \bar{e}_j) = \delta_{ij}$ $i, j = 1, \ldots, n$), then, because of (2.2), $\bar{D}_X Y$ is given by.

$$\bar{D}_X Y = \sum_{i=1}^{n} g(D_X{}^{\perp}D_Y{}^{\perp}\eta, D_{\bar{e}_i}{}^{\perp}\eta)\bar{e}_i. \quad (2.3)$$

For the Riemannian curvature of $\bar{U}$ in the two-dimensional directions spanned by $\bar{e}_r$ and $\bar{e}_k$ $(r \neq k)$ $r, k = 1, \cdots, n$, we find, because of (2.3), (2.2) and the fact that $\bar{D}$ is a metric connection:

$$\begin{aligned} \bar{K}(\bar{e}_r, \bar{e}_k) &= -\bar{g}(\bar{D}_{\bar{e}_r}\bar{D}_{\bar{e}_k}\bar{e}_r - \bar{D}_{\bar{e}_k}\bar{D}_{\bar{e}_r}\bar{e}_r - \bar{D}_{[\bar{e}_r, \bar{e}_k]}\bar{e}_r, \bar{e}_k) \\ &= -\bar{e}_r g(D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_k}{}^{\perp}\eta) + \sum_{i=1}^{n} g(D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_i}{}^{\perp}\eta)\bar{g}(\bar{e}_i, \bar{D}_{\bar{e}_r}\bar{e}_k) \\ &+ \bar{e}_k g(D_{\bar{e}_r}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_k}\eta) - \sum_{i=1}^{n} g(D_{\bar{e}_r}{}^{\perp}D_{\bar{e}_k}{}^{\perp}\eta, {}^{\perp}\eta)\bar{g}(\bar{e}_i, \bar{D}_{\bar{e}_k}\bar{e}_k) \\ &+ g(D_{[\bar{e}_r, \bar{e}_k]}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_k}{}^{\perp}\eta) \\ &= -g(D_{\bar{e}_r}D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_k}{}^{\perp}\eta) - g(D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_r}{}^{\perp}D_{\bar{e}_k}{}^{\perp}\eta) \\ &+ \sum_{i=1}^{n} g(D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_i}{}^{\perp}\eta)g(D_{\bar{e}_r}{}^{\perp}D_{\bar{e}_k}{}^{\perp}\eta, D_{\bar{e}_i}{}^{\perp}\eta) + g(D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_r}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_k}{}^{\perp}\eta) \end{aligned}$$

$$+g(D_{\bar{e}_r}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_k}{}^{\perp}\eta)-\sum_{i=1}^{n}g(D_{\bar{e}_r}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_i}{}^{\perp}\eta)g(D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_k}{}^{\perp}\eta, D_{\bar{e}_i}\eta)$$

$$+g(D_{[\bar{e}_r,\bar{e}_k]})^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, D_{\bar{e}_k}{}^{\perp}\eta). \tag{2.4}$$

The sum of the first, the fourth and the last part of this expression is equal to zero, because N has flat normal connection and $D_{\bar{e}_r}{}^{\perp}\eta$ is a normal vector field on N.

Next, since $\bar{g}(\bar{e}_i, \bar{e}_j)=g(D_{\bar{e}_i}{}^{\perp}\eta, D_{\bar{e}_j}{}^{\perp}\eta)=\delta_{ij}$ $i, j=1, \cdots, n$, the vector fields $D_{\bar{e}_i}{}^{\perp}\eta$ are orthonormal and we find also

$$0=\bar{e}_k\, g(D_{\bar{e}_r}{}^{\perp}\eta,\ \eta)=g(D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, \eta)$$

because $k\neq r$. So, $D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta$ and $D_{\bar{e}_r}{}^{\perp}D_{\bar{e}_k}{}^{\perp}\eta$ are orthogonal with η and moreover they belong to the subbundle $U_1{}^{\perp}$, because $U_1{}^{\perp}$ is parallel.

From all this we get that the sum of the second and the third part of the right side of (2.4) vanishes.

Finally, we get

$$0=\bar{e}_j g(D_{\bar{e}_j}{}^{\perp}\eta, \eta)=g(D_{\bar{e}_j}{}^{\perp}D_{\bar{e}_j}{}^{\perp}\eta, \eta)+g(D_{\bar{e}_j}{}^{\perp}\eta, D_{\bar{e}_j}{}^{\perp}\eta)\quad j=1, \cdots, n,$$

so

$$g(D_{\bar{e}_k}{}^{\perp}D_{\bar{e}_k}{}^{\perp}\eta, \eta)=g(D_{\bar{e}_r}{}^{\perp}D_{\bar{e}_r}{}^{\perp}\eta, \eta)=-1.$$

Because of this, we find in the same way as before, that the sum of the fifth and sixth part of the right side of (2.4) is equal to $+1$. So, for each two-dimensional direction spanned by $\bar{e}_r$ and $\bar{e}_k$, $r\neq k$, $r, k=1, \ldots, n$, we have Riemannian curvature $+1$ for $\bar{U}$ and this means that $\bar{U}$ has constant curvature $+1$.

REMARKS. 1. It is possible to proof this theorem in a shorter way using the property that a n-dimensional manifold N with flat normal connection in R^m admits locally $m-n$ orthonormal normal vector fields $\xi_1, \ldots, \xi_{m-n}$ such that each ξ_i is parallel in the normal bundle $N^{\perp}$. But we prefer the more general method involving the Riemannian connection $\bar{D}$ since we need $\bar{D}$ in the following two theorems.

2. If $m=2n+1$, then $U_1{}^{\perp}=U^{\perp}$ and $U_1{}^{\perp}$ is of course parallel.

3. If N is totally geodesic in R^m and R^m is a conformally flat space, then N has always flat normal connection, and every normal section is geodesic.

4. In fact it is sufficient to suppose that U has flat normal connection with respect to $U_1{}^{\perp}$ in order to have theorem 2.1. : by this we mean that $D_X{}^{\perp}D_Y{}^{\perp}\xi-D_Y{}^{\perp}D_X{}^{\perp}\xi-D_{[X,Y]}\xi=0$ for all vector fields X and Y of U and for each vector field ξ of $U_1{}^{\perp}$.

THEOREM 2. 2. *Under the same assumptions as for* theorem 2. 1., *we have that a curve* $\sigma : [a,b] \rightarrow \bar{U}$; $\bar{s} \rightarrow \sigma(\bar{s})$, *with* $\bar{U}$*-arc length* $\bar{s}$ *and with tangent vector field* $\bar{T}$, *is a geodesic of* $\bar{U}$, *iff* $D_{\bar{T}}{}^{\perp} D_{\bar{T}}{}^{\perp}\eta = -\eta$ *along* σ.

Proof. σ is a geodesic of $\bar{U}$ iff $\bar{D}_{\bar{T}}\bar{T}=0$ along σ or, because of (2. 3), iff

$$\sum_{i=1}^{n} g(D_{\bar{T}}{}^{\perp} D_{\bar{T}}{}^{\perp}\eta, D_{\bar{e}_i}{}^{\perp}\eta)\bar{e}_i = 0 \text{ along } \sigma.$$

So, $g(D_{\bar{T}}{}^{\perp} D_{\bar{T}}{}^{\perp}\eta, D_{\bar{e}_i}{}^{\perp}\eta)=0$, $i=1,\cdots,n$ and moreover

$$0=\bar{T}g(D_{\bar{T}}{}^{\perp}\eta,\eta)=g(D_{\bar{T}}{}^{\perp}D_{\bar{T}}{}^{\perp}\eta,\eta)+g(D_{\bar{T}}{}^{\perp}\eta, D_{\bar{T}}{}^{\perp}\eta).$$

But $\bar{g}(\bar{T},\bar{T})=g(D_{\bar{T}}{}^{\perp}\eta, D_{\bar{T}}{}^{\perp}\eta)=1$ and $D_{\bar{T}}{}^{\perp}D_{\bar{T}}{}^{\perp}\eta$ belongs to the normal subbundle $U_1{}^{\perp}$, thus $D_{\bar{T}}{}^{\perp}D_{\bar{T}}{}^{\perp}\eta=-\eta$.

THEOREM 2. 3. *Suppose that, with the same assumption as for* theorem 2. 1, $\sigma : [-a, +a] \rightarrow \bar{U}$; $\bar{s} \rightarrow \sigma(\bar{s})$ *is a geodesic of* $\bar{U}$, *with* $\bar{U}$*-arc length* $\bar{s}$ *and with tangent vector field* $\bar{T}$. *If* $\sigma(0)=p$, *then the parallel displacement in* R^m *of* η_p *along* σ *is locally given by*

$$\xi=\eta\cos\bar{s}-(D_{\bar{T}}{}^{\perp}\eta)\sin\bar{s}.$$

Proof. Suppose that ξ is the parallel vector field along σ with $\xi_p=\eta_p$. Consider

$$\alpha=g(\xi,\eta)\eta+g(D_{\bar{T}}{}^{\perp}\eta,\xi)D_{\bar{T}}{}^{\perp}\eta,$$

then we get

$$D_{\bar{T}}{}^{\perp}\alpha=(\bar{T}g(\xi,\eta))\eta+g(\xi,\eta)D_{\bar{T}}{}^{\perp}\eta+(\bar{T}g(D_{\bar{T}}{}^{\perp}\eta,\xi))D_{\bar{T}}{}^{\perp}\eta$$
$$+g(D_{\bar{T}}{}^{\perp}\eta,\xi)D_{\bar{T}}{}^{\perp}D_{\bar{T}}{}^{\perp}\eta$$

Since $D_{\bar{T}}{}^{\perp}\xi=0$ and because of the previous theorem, we find

$$D_{\bar{T}}{}^{\perp}\alpha=g(D_{\bar{T}}{}^{\perp}\xi,\eta)\eta+g(\xi,D_{\bar{T}}{}^{\perp}\eta)\eta+g(\xi,\eta)D_{\bar{T}}{}^{\perp}\eta+g(-\eta,\xi)D_{\bar{T}}{}^{\perp}\eta$$
$$+g(D_{\bar{T}}{}^{\perp}\eta,D_{\bar{T}}{}^{\perp}\xi)D_{\bar{T}}{}^{\perp}\eta-g(D_{\bar{T}}{}^{\perp}\eta,\xi)\eta=0,$$

and moreover we have $\alpha_p=\eta_p$, so $\alpha=\xi$. Consequently ξ is of the form $a\eta+bD_{\bar{T}}{}^{\perp}\eta$ and we get

$$D_{\bar{T}}{}^{\perp}\xi=(\bar{T}a)\eta+aD_{\bar{T}}{}^{\perp}\eta+(\bar{T}b)D_{\bar{T}}{}^{\perp}\eta+bD_{\bar{T}}{}^{\perp}D_{\bar{T}}{}^{\perp}\eta$$
$$=(\bar{T}a-b)\eta+(a+\bar{T}b)D_{\bar{T}}{}^{\perp}\eta=0.$$

Thus $\bar{T}a=b$, $\bar{T}b=-a$ or $\dfrac{da}{d\bar{s}}=b$, $\dfrac{db}{d\bar{s}}=-a$ with $a(0)=1$, $b(0)=0$,

so we have locally $a=\cos\bar{s}$ and $b=-\sin\bar{s}$, which completes the proof.

§ 3. Examples

Suppose that N is a submanifold of a complete simply connected space $R^m(c)$ of constant curvature c. We can consider $R^m(c)$ as a totally geodesic submanifold of an $R^{m+k}(c)$ $(k\geqslant 1)$ and every normal vector field η on N which is at each point p of his domain U orthogonal with $(R^m(c))_p$, is a geodesic vector field on N.

Assume that N is a submanifold of the euclidean space E^m and that ξ is any unit vector field (not necessarily normal on N) with domain $U\subset N$, such that $\tilde{D}_{X_p}\xi\neq 0$ for each $X_p\neq 0$ of N_p at each point p of U. Binding ξ at a fixed point of space, his endpoint describes the spherical image of N (or U) with respect to ξ (this is an n-dimensional submanifold of a unit $(m-1)$-dimensional sphere S^{m-1}) and we have the corresponding map $f: U\to S^{m-1}$; $p\to f(p)=$point with vector coordinate ξ_p. Then one can ask: what is a geometrical signification of the ratio of volume elements $\bar{\omega}$ and ω of the spherical image $f(U)$ at $f(p)$ and of U at p?

Consider $E^{2m}=E^m\times E^m$ with an orthonormal coordinate system $(x^1, \cdots, x^{2m})$, such that N is locally given by the parametric representation

$$x^i=f^i(u^1, \dots, u^n), \qquad i=1, \dots, m.$$
$$x^j=0, \qquad j=m+1, \dots, 2m.$$

and such that ξ has components $(\xi^1(u^1, \dots, u^n), \dots, \xi^m(u^1, \cdots, u^n), 0, \dots, 0)$. Construct the following $(n+1)$-dimensional submanifold C of E^{2m}:

$$x^i=f^i(u^1, \dots, u^n), \qquad i=1, \dots, m.$$
$$x^j=k\,\xi^{j-m}(u^1, \dots, u^n), \qquad j=m+1, \dots, 2m;\ k\in R.$$

It is clear that C contains N as a totally geodesic hypersurface (set $k=0$). Moreover, the spherical images constructed with the vector field ξ or the vector field ξ' with components $(0, \dots, 0, \xi^1(u^1, \dots, u^n), \dots, \xi^m(u^1, \dots, u^n))$ are isometric. Thus, from theorem 1. 1. 3°, we have at p : $(\bar{\omega}/\omega)^2=(-1)^n K(\xi_p)$, where $K(\xi_p)$ is the product of the n extremal values of the Riemann curvatures of C at p in the two-dimensional directions containing ξ_p'.

In fact, in $f: U\to S^{m-1}$ is any local immersion with domain $U\subset N$ of N into the unit $(m-1)$-sphere, the previous method method gives a geometrical signification of the ratio of volume elements $\bar{\omega}$ and ω of the immersed manifold $f(U)$ and resp. U at corresponding points $f(p)$ and p.

Special case: if N is a hypersurface in E^{n+1} and if ξ is the unit normal vector field on N in E^{n+1}, then it is easy to see that $K(\xi_p)=(-1)^n G^2(p)$, where $G(p)$ is the Gauss curvature of N at p. If $f: U\to f(U)\subset S^{m-1}$ is an isometric mapping, then we know from theorem 1. 1. 2° that C has at each point p of U in two-dimensional directions normal on U_p, constant curvature -1. In this connection we have the following property for an n-dimensional submanifold N of E^m : N has constant positive curvature iff for each point q of N we have a neighbourhood U on N and a straight line through each point q of U in $E^{m+n+1}=E^m \times E^{n+1}$, perpendicular on E^m (in which N lies), such that the $(n+1)$-dimensional manifold C generated by these lines has at each point q of U in each two-dimensional direction of C_q orthogonal on U_q, constant (negative) curvature.

Remark that we have in this first example that each vector field of $U_1^\perp$ is geodesic on U. Conversely, working in a space $R^m(c)$, if $U_1^\perp$ is parallel in $U^\perp$ and if moreover the connected domain U is geodesic with respect to any vector field of $U_1^\perp$, then U is contained in a totally geodesic submanifold $R^{m-n-1}(c)$ of $R^m(c)$ such that $U_1^\perp$ is everywhere perpendicular to $R^{m-n-1}(c)$ (see [4], p. 351).

Next, we give another example: consider a curve $\sigma :]a, b[\to E^5;\ u \to \sigma(u)$ with arc length u, curvature functions $\dfrac{1}{\rho_i(u)}$, $i=1, \dots, 4$ and Frenet-frame $(f_1(u), \dots, f_5(u))$. Assume that $\dfrac{1}{\rho_1(u)} \neq 0$ and $\dfrac{1}{\rho_3(u)} \neq 0$ for all $u \in]a, b[$. Consider the part of the tangent surface N of the curve given by

$$\sigma(u)+vf_1(u) \quad u \in]a, b[, \ v \in]0, \pi/2[,$$

on which we have the normal vector field $\eta=f_4 \cos v+f_5 \sin v$. It is easy to see, because of the Frenet formulas, that η is nonparallel and that N is geodesic with respect to η. Moreover, N has flat normal connection (because one-dimensional first normal space) in E^5, and thus, from theorem 2. 1., we know that $\bar{N}$ has constant curvature $+1$. We find after some calculations:

$$d\bar{s}^2=\left(\frac{\cos^2 v}{\rho_3^2}+\frac{1}{\rho_4^2}\right)du^2+\frac{2}{\rho_4}dudv+dv^2, \tag{3. 1}$$

and this is a form which gives indeed curvature $+1$. For the first fundamental form of N, we get:

$$ds^2=\left(1+\frac{v^2}{\rho_1^2}\right)du^2+2dudv+dv^2. \tag{3. 2}$$

From (3. 1), (3. 2) and because of theorem 1. 1. 3°, we find at a point $p(u, v)$ of N:

$$K(\eta_p)=\frac{\cos^2 v}{v^2} \frac{\rho_1^2(u)}{\rho_3^2(u)}.$$

Remark that because of the form of $d\bar{s}^2$, the curves $u=$constant on N (thus the tangents of the curve σ) are geodesics of $\bar{N}$; this follows also immediately from theorem 3. 3.

If we bind the vector η at a fixed point 0 of E^5, his endpoint describes the spherical image of N: this is a 2-dimensional submanifold of the unit four dimensional sphere S^4 with center 0, which has constant curvature $+1$. Remark that spherical image is a ruled surface of S^4, i. e. a surface generated by geodesic lines (parts of great circles) of S^4.

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