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ON HYPERSURFACES OF $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$

Dedicated to Professor Kentaro Yano on his seventieth birthday

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0. Introduction.

K. Yano and M. Okumura [12] defined the (f, g, u, v, λ)-structure induced on submanifolds of codimension 2 of an almost Hermitian manifold or hypersurfaces of an almost contact metric manifold. Also, Yano [7] introduced the (f, g, u, v, λ)-structure on $S^n\times S^n$ as a submanifold of codimension 2 of a $(2n+2)$-dimensional Euclidean space E^{2n+2} or hypersurfaces of a $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$ and derived many valuable properties.

In 1974, G. D. Ludden and Okumura [6] studied the so-called invariant hypersurface of $S^n\times S^n$, which suggest very useful methods representing an invariant hypersurfaces of $S^n\times S^n$.

The main purpose of the present paper is to study some characterizations of hypersurfaces of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

In §1, we discuss the intrinsic properties of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

In §2, we find some algebraic relations and structure equations of hypersurfaces of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

In §3, we study a hypersurface M of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ admitting an almost contact metric structure and characterize a hypersurface M of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ as a submanifold of codimension 3 of an even-dimensional Euclidean space E^{2n+2}.

§4 is devoted to determine the hypersurface M as a submanifold of codimension 2 of an odd-dimensional unit sphere $S^{2n+1}(1)$. For later uses, we introduce the following theorems.

THEOREM A([11]). *Let M be a pseudo-umbilical submanifold of an even-dimensional Euclidean space E with (f, g, u, v, λ)-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. Then M is an intersection of a complex cone with generator as a normal vector and a sphere.*

According to the similar method of Ludden and Okumura [6], we obtain

THEOREM B. *Let M be a compact orientable totally geodesic invariant hyper-*

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surface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$. *Then M is* $S^{n-1}(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

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§1. Preliminaries.

Let E^{n+1} be an $(n+1)$-dimensional Euclidean space and O the origin of the Cartesian coordinate system in E^{n+1}, and denote by X the position vector of a point P_1 in E^{n+1} with respect to the origin.

We consider a sphere $S^n(1/\sqrt{2})$ with center at O and radius $1/\sqrt{2}$ and suppose that $S^n(1/\sqrt{2})$ is covered by a system of coordinate neighborhoods $\{U\,;\,x^\alpha\}$. Here and in the sequel, the indices $\alpha,\beta,\gamma,\delta,\ldots$ run over the range $\{1,2,\ldots,n\}$. Then $X\cdot X=1/2$ for the position vector X at each point in $S^n(1/\sqrt{2})$, where the dot denotes the inner porduct of two vectors in a Euclidean space. Now we put $X_\alpha=\partial_\alpha X$, $M_1=-\sqrt{2}X$, $g_{\alpha\beta}=X_\alpha\cdot X_\beta$ where $\partial_\alpha=\partial/\partial x^\alpha$, and denote by ∇_α the operator of covariant differentiation formed with the metric tensor $g_{\alpha\beta}$. Then the equations of Gauss and Weingarten are respectively given by

$$(1.1)\qquad \nabla_\beta X_\alpha=\sqrt{2}\,g_{\beta\alpha}M_1,\qquad \nabla_\alpha M_1=-\sqrt{2}\,X_\alpha.$$

We next suppose that $S^n(1/\sqrt{2})$ is also covered by a system of coordinate neighborhoods $\{V\,;\,x^\kappa\}$. Then the position vector Y of a point on $S^n(1/\sqrt{2})$ satisfies $Y\cdot Y=1/2$. Here and in the sequel, the indices $\kappa,\mu,\nu,\ldots$ run over the range $\{n+1,\ldots,2n\}$. We put $Y_\kappa=\partial_\kappa Y$, $M_2=-\sqrt{2}\,Y$, $g_{\kappa\mu}=Y_\kappa\cdot Y_\mu$, where $\partial_\kappa=\partial/\partial x^\kappa$ and denote by ∇_κ the operator of covariant differentiation formed with the metric tensor $g_{\kappa\mu}$ of $S^n(1/\sqrt{2})$. Then the equations of Gauss and Weingarten are respectively of the form

$$(1.2)\qquad \nabla_\kappa Y_\mu=\sqrt{2}\,g_{\kappa\mu}M_2,\qquad \nabla_\kappa M_2=-\sqrt{2}\,Y_\kappa.$$

If we regard $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ as a submanifold of codimension 2 in a $(2n+2)$-dimensional Euclidean space E^{2n+2}, then we have a position vector Z of a point of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ such that

$$Z(x^h)=\begin{pmatrix}X(x^\kappa)\\ Y(x^\kappa)\end{pmatrix},$$

where, here and in the sequel, the indices $h,i,j,k,\ldots$ run over the range $\{1,2,\ldots,n,\ n+1,\ldots,2n\}$. Consequently, from the fact that $Z\cdot Z=X\cdot X+Y\cdot Y=1$, we see that $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ is a hypersurface of $S^{2n+1}(1)$ in E^{2n+2}.

Putting $Z_i=\partial_i Z$ and $g_{ji}=Z_j\cdot Z_i$, we see that

$$Z_\alpha=\begin{pmatrix} X_\alpha \\ O \end{pmatrix}, \qquad Z_\kappa=\begin{pmatrix} O \\ Y_\kappa \end{pmatrix}$$

and

$$g_{ji}=\begin{pmatrix} g_{\alpha\beta} & O \\ O & g_{\kappa\mu} \end{pmatrix}, \quad g^{ji}=\begin{pmatrix} g^{\alpha\beta} & O \\ O & g^{\kappa\mu} \end{pmatrix}, \tag{1.3}$$

where g^{ji}, $g^{\alpha\beta}$ and $g^{\kappa\mu}$ is elements of the inverse matrices of (g_{ji}), $(g_{\alpha\beta})$ and $(g_{\kappa\mu})$ respectively.

Now putting

$$C=\begin{pmatrix} -X(x^\alpha) \\ -Y(x^\kappa) \end{pmatrix}, \qquad D=\begin{pmatrix} -X(x^\alpha) \\ Y(x^\kappa) \end{pmatrix}, \tag{1.4}$$

we see that

$$Z_j\cdot C=0, \quad Z_j\cdot D=0, \quad C\cdot C=1, \quad D\cdot D=1$$

and consequently C and D are mutually orthogonal unit normals to $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

Let h_{ji} and k_{ji} be the components of the second fundamental tensors respectively with respect to C and D. Then we can write the equation of Gauss for $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ as

$$\nabla_j Z_i=h_{ji}C+k_{ji}D.$$

From (1.1) and (1.2) it follows that h_{ji} and k_{ji} are of the form

$$h_{ji}=\begin{pmatrix} g_{\alpha\beta} & O \\ O & g_{\kappa\mu} \end{pmatrix}, \quad k_{ji}=\begin{pmatrix} g_{\alpha\beta} & O \\ O & -g_{\kappa\mu} \end{pmatrix} \tag{1.5}$$

and hence

$$h_j{}^i=\begin{pmatrix} \delta_\alpha{}^\beta & O \\ O & \delta_\kappa{}^\mu \end{pmatrix}, \quad k_j{}^i=\begin{pmatrix} \delta_\alpha{}^\beta & O \\ O & -\delta_\kappa{}^\mu \end{pmatrix} \tag{1.6}$$

respectively, where $h_j{}^i=h_{jh}g^{ih}$ and $k_j{}^i=k_{jh}g^{ih}$.

The first equation of (1.5) and the second equation of (1.6) imply immediately that

$$h_{ji}=g_{ji}, \qquad k_j{}^j=0, \qquad k_j{}^t k_t{}^i=\delta_j^i. \tag{1.7}$$

The second equation of (1.7) shows that $k_j{}^i$ defines an almost product structure of $S^n\times S^n$.

On the other hand the Christoffel symbols $\{_j{}^h{}_i\}$ are all zero except $\{_\gamma{}^\alpha{}_\beta\}$ and $\{_\kappa{}^\lambda{}_\mu\}$ because of (1.3). Therefore, taking account of the fact that $k_j{}^i$ has the form given by the second equation of (1.6), we find

$$\nabla_j k_i{}^h=0.$$

Denoting by l_j the third fundamental tensor with respect to C and D we can write the equations of Weingarten as

$$(1.8)\qquad \nabla_j C=-h_j{}^t Z_t+l_j D,\qquad \nabla_j D=-k_j{}^t Z_t-l_j C.$$

From (1.4), (1.6) and (1.8) it follows that $l_j=0$. Thus the equations of Gauss and Weingarten of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ as a submanifold of codimension 2 of E^{2n+2} are respectively

$$\nabla_j Z_i=g_{ji}C+k_{ji}D,\qquad \nabla_j C=-Z_j,\qquad \nabla_j D=-k_j{}^t Z_t,$$

from which we can easily derive

$$(1.9)\qquad K_{kji}{}^h=\delta_k^h g_{ji}-\delta_j^h g_{ki}+k_k{}^h k_{ji}-k_j{}^h k_{ki}$$

which are the equations of Gauss, $K_{kji}{}^h$ being the components of the curvature tensor of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

In E^{2n+2}, there exists a natural Kaehlerian structure $F=\begin{pmatrix} O & -E\\ E & O\end{pmatrix}$, E being the unit matrix of degree $n+1$. It follows that $F^2=-I$, $FU\cdot FV=U\cdot V$ for arbitrary vectors U and V in E^{2n+2}, I denoting the identity transformation in E^{2n+2}. Transforming Z_j, C and D by F, we have

$$(1.10)\qquad FZ_j=f_j{}^t Z_t+u_j C+v_j D,\qquad FC=-u^t Z_t+\lambda D,\qquad FD=-v^t Z_t-\lambda C,$$

where $f_i{}^h$ are the components of a tensor field of type (1, 1), u_i and v_i are those of 1-forms and λ is a function on $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$, u^h and v^h are respectively given by $u^h=u_t g^{th}$ and $v^h=v_t g^{th}$.

Applying F to (1.10) respectively, we get the so-called (f, g, u, v, λ)-structure given by (see [8], [9] and [10])

$$(1.11)\qquad \begin{cases} f_j{}^t f_t{}^i=-\delta_j^i+u_j u^i+v_j v^i,\\ u_t f_j{}^t=\lambda v_j,\qquad f_t{}^h u^t=-\lambda v^h,\qquad v_t f_j{}^t=-\lambda u_j,\qquad f_t{}^h v^t=\lambda u^h,\\ u_t u^t=v_t v^t=1-\lambda^2,\qquad u_t v^t=0,\\ f_j{}^t f_i{}^s g_{ts}=g_{ji}-u_j u_i-v_j v_i.\end{cases}$$

It is easily verified that $f_{ji}=f_j{}^h g_{ih}$ is skew-symmetric in j and i.

By putting $j=\alpha$ in (1.10), we find

$$(1.12)\qquad f_\alpha{}^\beta=0,\qquad u_\alpha+v_\alpha=0,\qquad X_\alpha=f_\alpha{}^\kappa Y_\kappa-2u_\alpha Y.$$

In the same way, putting $j=\kappa$ in (1.10), we find

$$(1.13)\qquad f_\kappa{}^\mu=0,\quad u_\kappa=v_\kappa,\quad Y_\kappa=-f_\kappa{}^\alpha X_\alpha-2u_\kappa X.$$

Thus $f_i{}^h, u_i, u^h, v_i$ and v^h are respectively of the form

$$(1.14)\qquad f_i{}^h=\begin{pmatrix} O & f_\kappa{}^\alpha \\ f_\alpha{}^\nu & O \end{pmatrix},$$

$$(1.15)\qquad u_i=(u_\alpha, u_\kappa),\qquad u^h=\begin{pmatrix} u^\alpha \\ u^\kappa \end{pmatrix},\qquad v_i=(-u_\alpha, u_\kappa),\qquad v^h=\begin{pmatrix} -u^\alpha \\ u^\kappa \end{pmatrix},$$

where $u^\alpha=u_\beta g^{\alpha\beta}$, $u^\kappa=u_\mu g^{\kappa\mu}$.

From (1.6) and (1.14), it follows that

$$(1.16)\qquad k_t{}^h f_j{}^t+f_t{}^h k_j{}^t=0,$$

that is, $k_j{}^h$ and $f_i{}^h$ anticommute each other. We also find from (1.6) and (1.15)

$$(1.17)\qquad k_t{}^h u^t=-v^h,\qquad k_t{}^h v^t=-u^h.$$

Now, differentiating (1.10) covariantly and using $\nabla F=0$, we obtain

$$(1.18)\qquad \begin{cases} \nabla_j f_i{}^h=-g_{ji}u^h+\delta_j^h u_i-k_{ji}v^h+k_j{}^h v_i, \\ \nabla_j u_i=f_{ji}-\lambda k_{ji},\qquad \nabla_j v_i=-k_{jt}f_i{}^t+\lambda g_{ji},\qquad \nabla_j\lambda=-2v_j. \end{cases}$$

§2. Structure equations of hypersurfaces of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

Let M be a hypersurface immersed isometrically in $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ by the immersion $i: M\longrightarrow S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ and suppose that M is covered by a system of coordinate neighborhoods $\{\bar{V}\,;\, y^a\}$. Here and in the sequel, the indices $a, b, c, d, \ldots$ run over the range $\{1, 2, \ldots, 2n-1\}$.

We put

$$B_c{}^h=\partial_c x^h,\qquad \partial_c=\partial/\partial y^c.$$

Then $B_c{}^h$ are $2n-1$ linearly independent vectors of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ tangent to M. And denote by N^h the unit normal vector to M. Then, denoting by g_{cb} the components of the induced metric tensor of M, we have $g_{cb}=g_{ji}B_c{}^j B_b{}^i$ since the immersion is isometric.

As to transform of $B_c{}^j$ and N^j by $f_j{}^h$, we have respectively

(2.1)
$$\begin{cases} f_j{}^h B_c{}^j = f_c{}^a B_a{}^h + w_c N^h, \\ F_j{}^h N^j = -w^a B_a{}^h, \end{cases}$$

where $f_c{}^a$ denotes the components of a tensor field of type (1,1), w_c 1-form and w^a vector field associated with w_c such that $w^a = w_c g^{ac}$, g^{ac} being the contravariant components of the induced metric tensor g_{cb}.

We may put the vector fields u^h and v^h as follows

(2.2)
$$u^h = u^a B_a{}^h + \mu N^h, \qquad v^h = v^a B_a{}^h + \nu N^h,$$

where u^a, v^a are vector fields, μ and ν are certain functions on M.

Applying $f_h{}^k$ to (2.1) and (2.2) respectively, and using (1.10), we obtain the so-called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure as follows:

(2.3)
$$f_b{}^e f_e{}^a = -\delta_b{}^a + u_b u^a + v_b v^a + w_b w^a,$$

(2.4)
$$\begin{cases} f_e{}^a u^e = -\lambda v^a + \mu w^a, \\ f_e{}^a v^e = \lambda u^a + \nu w^a, \\ f_e{}^a w^e = -\mu u^a - \nu v^a, \end{cases}$$

or,

$$u_e f_c{}^e = \lambda v_c - \mu w_c, \qquad v_e f_c{}^e = -\lambda u_c - \nu w_c, \qquad w_e f_c{}^e = \mu u_c + \nu v_c,$$

(2.5)
$$\begin{cases} u_e u^e = 1 - \lambda^2 - \mu^2, & u_e v^e = -\mu\nu, & u_e w^e = -\lambda\nu, \\ & v_e v^e = 1 - \lambda^2 - \nu^2, & v_e w^e = \lambda\mu, \\ & & w_e w^e = 1 - \mu^2 - \nu^2, \end{cases}$$

where u_c, v_c and w_c are 1-forms associated with u^a, v^a and w^a respectively.

We can easily verified that $f_{cb} = f_c{}^a g_{ba}$ are skew-symmetric because f_{ji} is skew-symmetric. Transvecting the last equation of (1.11) with $B_c{}^j B_b{}^i$ and making use of (2.2) and the definition of the induced metric tensor g_{cb}, we find

$$f_c{}^e f_b{}^d g_{ed} = g_{cb} - u_c u_b - v_c v_b - w_c w_b.$$

We now put

(2.6)
$$\begin{cases} k_j{}^h B_b{}^j = k_b{}^a B_a{}^h + k_b N^h, \\ k_j{}^h N^j = k^a B_a{}^h + \alpha N^h, \end{cases}$$

where $k_b{}^a$ are the components of a tensor field of type (1,1), k_b 1-form, k^a vector field associated with k_b and α some function.

When the action of the tangent space is invariant under the tensor field

$k_j{}^i$, that is, $k_c=0$, we call the hypersurface M is invariant if M is orientable.

Transvecting the first equation of (2.6) with $B^c{}_k=B_b{}^j g^{cb} g_{jk}$, and contracting h and k, we find

(2.7) $$k_e{}^e=-\alpha.$$

Applying $k_j{}^h$ to (2.6) respectively and taking account of (1.7) and these equations, we find

(2.8) $$k_c{}^e k_e{}^a=\delta_c^a-k_c k^a,$$

(2.9) $$k_c{}^e k_e=-\alpha k_c,$$

(2.10) $$k_e k^e=1-\alpha^2.$$

Transvecting (2.6) with $f_j{}^h$ and using (1.16), (2.1) and (2.6) itself, we have

(2.11) $$k_c{}^e f_e{}^a+f_c{}^e k_e{}^a=k_c w^a-w_c k^a,$$

(2.12) $$k_c{}^e w_e+f_c{}^e k_e=-\alpha w_c.$$

If we make use of (1.17), (2.2) and (2.6), we get

(2.13) $$\begin{cases} k_c{}^e v_e=-u_c-\nu k_c, & k_e v^e=-\mu-\alpha\nu, \\ k_c{}^e u_e=-v_c-\mu k_c, & k_e u^e=-\nu-\alpha\mu. \end{cases}$$

If we denote by ∇_c the operator of the van der Waerden-Bortolotti covariant differentiation, we can write the equations of Gaus and those of Weingarten respectively of the form

(2.14) $$\nabla_c B_b{}^h=l_{cb}N^h, \qquad \nabla_c N^h=-l_c{}^a B_a{}^h,$$

where l_{cb} denotes the second fundamental tensor with respect to the unit normal N^h and $l_c{}^a=g^{ab}l_{cb}$. Then, from (1.9), the equation of Gauss is given by

$$K_{dcb}{}^a=\delta_d^a g_{cb}-\delta_c^a g_{db}+k_d{}^a k_{cb}-k_c{}^a k_{db}+l_d{}^a l_{cb}-l_c{}^a l_{db},$$

where we have put $k_{cb}=k_c{}^a g_{ab}$.

Differentiating (2.1), (2.2) and (2.6) covariantly along M, and using (1.18), (2.14) and the fact that $\nabla_j k_i{}^h=0$, we obtain the following structure equations:

(2.15) $$\nabla_c f_b{}^a=-g_{cb}u^a+\delta_c^a u_b-k_{cb}v^a+k_c{}^a v_b-l_{cb}w^a+l_c{}^a w_b,$$

(2.16)
$$\begin{cases} \nabla_c u_b = \mu l_{cb} - \lambda k_{cb} + f_{cb}, \\ \nabla_c v_b = k_c{}^e f_{eb} - k_c w_b + \nu l_{cb} + \lambda g_{cb}, \\ \nabla_c w_b = -\mu g_{cb} - \nu k_{cb} + k_c v_b - l_{ce} f_b{}^e, \end{cases}$$

(2.17)
$$\nabla_c \lambda = -2v_c, \qquad \nabla_c \mu = w_c - \lambda k_c - l_{ce} u^e, \qquad \nabla_c \nu = k_{ce} w^e - l_{ce} v^e,$$

(2.18)
$$\nabla_c k_b{}^a = l_{cb} k^a + l_c{}^a k_b,$$

(2.19)
$$\nabla_c k_b = -k_{ba} l_c{}^a + \alpha l_{cb},$$

(2.20)
$$\nabla_c \alpha = -2 l_{ce} k^e.$$

If we consider M as a submanifold of codimension 2 of $S^{2n+1}(1)$, we easily find from above structure equations that k_c is the third fundamental tensor.

§3. A hypersurface M of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ admitting an almost contact metric structure.

In this section we assume that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on M satisfies $\lambda^2 + \mu^2 + \nu^2 = 1$ and $n > 1$.

Computing the square of the length of $\mu u_b + \nu v_b$, we find

(3.1)
$$\mu u_b + \nu v_b = 0$$

because of (2.5) and the assumption $\lambda^2 + \mu^2 + \nu^2 = 1$, and consequently $w_e f_b{}^e$ vanishes identically because of (2.4). In the same manner we can see that $u_e f_b{}^e$ and $v_e f_b{}^e$ vanish identically on M.

LEMMA 3.1. *Let M be a hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ $(n > 1)$ with the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$. Then we have*

(3.2)
$$\mu = 0,$$

(3.3)
$$v_b = 0,$$

(3.4)
$$\nu = \text{const.}\ (\neq 0).$$

Proof. First of all we verify that the function ν is not identically zero on any open set $\tilde{O}$ in M. In fact, if ν vanishes on $\tilde{O}$, we have from the first equation of (2.5), $u_b = 0$ because $1 - \lambda^2 - \mu^2 = 0$. This and the first equation of (2.16) show that

$$\mu l_{cb} - \lambda k_{cb} + f_{cb} = 0,$$

which implies that $f_{cb} = 0$ because l_{cb} and k_{cb} are symmeric and f_{cb} is skew-symmetric with respect to b and c. Contracting (2.3) with respect to a

and b, we obtain $n=1$ with the help of (2.5). It contradicts $n>1$. Therefore, the function ν does not vanish on any open set in M. This means that ν takes nonzero value at some point in M.

Next we differentiate (3.1) covariantly and take the skew-symmetric part and get

$$(\nabla_c\mu)u_b-(\nabla_b\mu)u_c+\mu(f_{cb}-f_{bc})+(\nabla_c\nu)v_b-(\nabla_b\nu)v_c$$
$$+\nu(f_{eb}k_c{}^e-k_cw_b-f_{ec}k_b{}^e+k_bw_c)=0,$$

because of (2.16). Using (2.11), we find

$$(\nabla_c\mu)u_b-(\nabla_b\mu)u_c+2\mu f_{cb}+(\nabla_c\nu)v_b-(\nabla_b\nu)v_c=0.$$

Transvecting f^{cb} to the last equation and making use of the fact that $f_e{}^au^e=f_e{}^av^e=0$, we have

$$2\mu f_{cb}f^{cb}=0.$$

Thus we have (3.2). (3.3) is an easy consequence from (2.5) and (3.2) because of $\lambda^2+\mu^2+\nu^2=1$. To prove (3.4) we use the first equation of (2.17) which shows that $\lambda=$const.. Hence $\nu=\pm\sqrt{1-\lambda^2-\mu^2}=\pm\sqrt{1-\lambda^2}=$const. ν takes nonzero at some point of M. Therefore, ν is nonzero constant. This completes the proof of Lemma 3.1.

THEOREM 3.2 *Let M be a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ $(n>1)$ with the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. Then M is an almost contact metric manifold with structure $(f_a{}^b, g_{cb}, k_a)$.*

Proof. We have from the second equation of (2.13) and (3.4) that

$$\alpha=0, \tag{3.5}$$

from which, together with (2.10),

$$k_ek^e=1. \tag{3.6}$$

Moreover, the first equation of (2.13), we have

$$u_c=-\nu k_c. \tag{3.7}$$

Substituting (3.7) into the second equation of (2.4) and using (3.3), we get

$$w_c=\lambda k_c \tag{3.8}$$

because of (3.4). Hence it follows that

$$f_b{}^ef_e{}^a=-\delta_b{}^a+k_bk^a,$$

or, equivalently,

$$f_c{}^d f_b{}^e g_{de} = g_{cb} - k_c k_b.$$

These, together with (3. 6), imply that $(f_b{}^a, g_{cb}, k_a)$ defines an almost contact metric structure on M.

Now we consider such a hypersurface M of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ that v^h is the unit normal vector field. Then we may put $v^h = -N^h$. This assumption implies that $\lambda = 0$ and $\nu = -1$.

Differentiating (3. 7) and (3. 8) covariantly and using (2. 16), (2. 19), (3. 2), (3. 3) and (3. 5), we have

$$f_{cb} = -l_{ce} k_b{}^e, \tag{3. 9}$$

$$k_{cb} = l_{ce} f_b{}^e \tag{3. 10}$$

respectively. Since $\nu = -1$, we have from (3. 7)

$$u_b = k_b. \tag{3. 11}$$

Differentiating this covariantly, we find

$$\nabla_c k_b = f_{cb} \tag{3. 12}$$

with the aid of (2. 16), (2. 19), (3. 2), (3. 3) and (3. 5). Substituting (3. 11) into (2. 15) and making use of (3. 3) and (3. 8) with $\lambda = 0$, we have

$$\nabla_c f_b{}^a = -g_{cb} k^a + \delta_c^a k_b. \tag{3. 13}$$

If we transvect (3. 9) with k^{cb} and use (2. 8), (2. 20) with $\alpha = 0$, then we find

$$l_e{}^e = 0. \tag{3. 14}$$

On the other hand, if we transvect (3. 9) with $k_d{}^b$, we get

$$l_{cb} = -f_{ce} k_b{}^e. \tag{3. 15}$$

Transvecting this with $l_d{}^c$ and taking account of (3. 10), we have

$$l_{de} l_b{}^e = k_{de} k_b{}^e = g_{db} - k_d k_b \tag{3. 16}$$

with the help of (2. 8). Making use of (3. 5), (3. 14) and (3. 16), the Ricci tensor is given by

$$K_{cb} = 2(n-2) g_{cb} + 2 k_c k_b.$$

Therefore, from Theorem 3. 2, (3. 12), (3. 13), (3. 14) and above, we get

THEOREM 3. 3. *Let M be a hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ $(n>1)$ with the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$. If we take v^h as the unit normal vector, then M is a minimal Sasakian C-Einstein hypersurface of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.*

Consider a diagram:

$$M \xrightarrow{i} S^n(1/\sqrt{2})\times S^n(1/\sqrt{2}) \xrightarrow{\tilde{i}} E^{2n+2},$$

where i and $\tilde{i}$ are imbeddings. We now put the tangent vectors Z_a and the normal vector N of M as a submanifold of codimension 3 of a Euclidean space E^{2n+2} of dimension $2n+2$ by

$$Z_a=Z_jB_a{}^j, \qquad N=N^jZ_j.$$

Then we can see that l_{cb}, h_{cb} and k_{cb} are the second fundamental tensors with respect to the mutually orthogonal unit normal vector fields N, C and D respectively, where we have put $h_{cb}=h_{ji}B_c{}^jB_b{}^i$. And consequently it follows from (1.7) that

$$h_{cb}=g_{cb}. \tag{3.17}$$

Thus (2.7), (3.5), (3.14) and (3.17) imply that M is pseudo-umbilical ([11]). Hence, by taking account of Theorem A in §0, we have

THEOREM 3.4. *Let M be a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ $(n>1)$ with the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If we take v^h as the unit normal vector, then M as a submanifold of codimension 3 of a Euclidean space E^{2n+2} is an intersection of a complex cone with generator C and a $(2n+1)$-dimensional sphere $S^{2n+1}(1)$.*

§4. Some characterizations of a hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

In this section we suppose that the hypersurface M of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ is compact orientable.

If we regard M as a submanifold of codimension 2 of $S^{2n+1}(1)$, then we have the equations of Codazzi

$$\nabla_d k_{cb}-\nabla_c k_{db}=k_c l_{db}-k_d l_{cb}, \tag{4.1}$$

$$\nabla_d l_{cb}-\nabla_c l_{db}=k_d k_{cb}-k_c k_{db}. \tag{4.2}$$

We now suppose that $k_c{}^e f_e{}^a+f_c{}^e k_e{}^a=0$ holds or, equivalently

$$k_{ce}f_b{}^e-k_{be}f_c{}^e=0, \tag{4.3}$$

and the function $\mu(1-\lambda^2-\mu^2-\nu^2)$ does not vanish almost everywhere. Then we have

$$k_c w_b-w_c k_b=0$$

because of (2.11), which shows that

(4. 4) $$(1-\mu^2-\nu^2)k_c=(k_aw^a)w_c, \qquad (1-\alpha^2)w_c=(k_aw^a)k_c$$

with the help of (2. 5) and (2. 10).

Transvecting (4. 3) with $k_d{}^c$ and taking account of (2. 8), we obtain

$$f_{bd}-k_d(k_ef_b{}^e)-k_{be}k_d{}^cf_c{}^e=0,$$

from which, taking the symmetric part with respect to the indices b and d,

(4. 5) $$k_d(f_{be}k^e)+k_b(f_{de}k^e)=0.$$

If we transvect w^d to (4. 5) and use (2. 4), we get

$$(k_aw^a)(f_{be}k^e)+k_b(-\mu u_e-\nu v_e)k^e=0,$$

or, use (2. 13)

(4. 6) $$(k_aw^a)(f_{be}k^e)+\{2\mu\nu+\alpha(\mu^2+\nu^2)\}k_b=0.$$

On the other hand, transvecting (4. 4) with f^{cb}, we find

$$f^{cb}k_{ce}f_b{}^e=0$$

because f^{cb} is skew-symmetric and $k_{ce}f_b{}^e$ is symmetric, from which, taking account of (2. 3), (2. 4), (2. 5), (2. 7), (2. 12) and (2. 13),

(4. 7) $$2\mu\nu+\alpha(\mu^2+\nu^2)=0.$$

Thus, (4. 6) reduces to

(4. 8) $$(k_aw^a)(f_{be}k^e)=0.$$

If $f_{be}k^e$ is not zero at some point P of M, then $k_aw^a=0$, which shows from (2. 10) and (4. 4) that $(1-\alpha^2)w_c=0$. Thus w_c is zero. It contradicts to the fact that $f_{be}k^e(P)\neq 0$. In the sequel, we have

(4. 9) $$f_{be}k^e=0.$$

Transvecting this with $f_a{}^b$ and taking account of (2. 3) and (2. 13), we get

(4. 10) $$k_a=(k_ew^e)w_a-(\mu+\alpha\nu)v_a-(\nu+\alpha\mu)u_a,$$

or, using (4. 4),

(4. 11) $$(\mu^2+\nu^2)k_a+(\mu+\alpha\nu)v_a+(\nu+\alpha\mu)u_a=0.$$

Transvecting (4. 11) with u^a and using (2. 5) and (2. 13), we find

$$(\nu+\alpha\mu)(1-\lambda^2-\mu^2-\nu^2)-\mu\{\alpha(\mu^2+\nu^2)+2\mu\nu\}=0,$$

or, using (4. 7) and the assumption that $\mu(1-\lambda^2-\mu^2-\nu^2)$ does not vanish almost everywhere, (4.12) $$\nu+\alpha\mu=0.$$

In the same way, if we transvect (4.11) with v^a, we get

(4.13) $$\mu+\alpha\nu=0.$$

Since $\mu(1-\lambda^2-\mu^2-\nu^2)$ does not vanish almost everywhere, we can see from (4.12) and (4.13)

(4.14) $$\mu^2=\nu^2,$$

and hence $1-\alpha^2=0$, that is,

(4.15) $$k_a=0$$

because of (2.10). This means that M is an invariant submanifold with respect to $k_j{}^i$. From (2.12) and (4.9) it follows that

(4.16) $$k_{ce}w^e=-\alpha w_c.$$

Since we see from (2.19) and (4.15)

(4.17) $$l_{ce}k_b{}^e=\alpha l_{cb},$$

we obtain

(4.18) $$l_{ce}w^e=0$$

by transvecting (4.17) with w^b and using (4.16).

Now, we assume that $l_c{}^e f_e{}^a+f_c{}^e l_e{}^a=0$, that is, $l_{ce}f_a{}^e-l_{ae}f_c{}^e=0$.

Differentiating (4.18) covariantly and taking account of (2.16), then we have

$$(\nabla_b l_{ce})w^e+l_c{}^e(-\mu g_{be}-\nu k_{be}-l_{ba}f_e{}^a)=0$$

because of (4.15), from which, taking the skew-symmetric part with respect to the indices b and c and using (4.2) and (4.15),

$$\nu(l_b{}^e k_{ce}-l_c{}^e k_{be})-l_c{}^e l_{ba}f_e{}^a+l_b{}^e l_{ca}f_e{}^a=0,$$

or using (4.17),

(4.19) $$l_c{}^a l_{ae}f_b{}^e=0$$

because of $l_c{}^e f_e{}^a+f_c{}^e l_e{}^a=0$.

If we transvect this with $f_d{}^b$ and use (2.3) and (4.18), then we get

(4.20) $$l_{ce}l_b{}^e=l_c{}^a\{(l_{ae}u^e)u_b+(l_{ae}v^e)v_b\}.$$

Differentiating (4.14) covariantly, and using (2.17), (4.15), (4.16) and above assumption, we have $\mu(w_c-l_{ce}u^e)=\nu(-\alpha w_c-l_{ce}v^e)$, or, using (4.13).

(4.21) $$\mu l_{ce}u^e=\nu l_{ce}v^e.$$

If we transvect (4.19) with w^b and take account of (2.4) and (4.21), then we have

$$(4.22) \qquad l_c{}^a l_{ae} u^e = 0, \qquad l_c{}^a l_{ae} v^e = 0.$$

Substituting (4.22) into (4.20), we obtain $l_{cb}=0$, that is, M is totally geodesic. Therefore, taking account of Theorem B in §0, we have

THEOREM 4.1. *Let M be a compact orientable hypersurface of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$. If $k_c{}^e f_e{}^a + f_c{}^e k_e{}^a = 0$, $l_c{}^e f_e{}^a + f_c{}^e l_e{}^a = 0$ and $\mu(l-\lambda^2-\mu^2-\nu^2)$ does not vanish almost everywhere, then M is $S^{n-1}(1/\sqrt{2})\times S^n(1/\sqrt{2})$.*

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