J. Korean Math. Soc.
Vol. 18, No. 2, 1982

GENERALIZED FUNDAMENTAL GROUPS OF CONTINUOUS LOOPS

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1. Introduction

Let Y be a topological space and let $y_0 \in Y$. Then $C(Y, y_0)$ will be used to denote the set of all continuous loops in Y at y_0. The idea of using continuous functions as relating functions on $C(Y, y_0)$ to get an equivalence relation on $C(Y, y_0)$ has long been in existence, and extensive studies have been made of the resulting homotopy groups.

In this paper, an admitting homotopy relation is defined, which in general, turns out to be a larger class of relating functions than the class of all continuous functions. This in turn leads to possibly more loops in each equivalence class and thus to fewer equivalence classes. An admitting homotopy relation is then used to obtain a generalized homotopy group which is usually smaller than the fundamental group [see Theorem 4].

Most types of non-continuous functions, including almost continuous functions [2], c-continuous functions [3], and connectivity mapt [17], provide an admitting homotopy relation.

2. Admitting homotopy relation

DEFINTION 1. Let N be a class of functions with the following four properties:

(i) N contains the class of all continuous functions,

(ii) if X, Y, and Z are topological spaces, $f : X \to Y$ is in N and $g : Y \to Z$ is a homeomorphism, then $gf : Y \to Z$ is in N,

(iii) if X, Y, and Z are topological spaces, $f : X \to Y$ is a homeomorphism, and $g : Y \to Z$ is in N, then $gf : X \to Z$ is in N, and

(iv) if a, b, c, d, α, and β are numbers such that $a < b < c < d$ and $\alpha < \beta$ and if $[a,d]\times[\alpha,\beta]$, $[a,b]\times[\alpha,\beta]$, $[c,d]\times[\alpha,\beta]$, and $[b,c]\times[\alpha,\beta]$ have the relative topology induced from the usual topology on the plane, and if $f : [a,d]\times[\alpha,\beta] \to Y$ is a function such that $f|_{[a,b]\times[\alpha,\beta]}$ and $f|_{[c,d]\times[\alpha,\beta]}$

Received June 10, 1981.

are in N and $f\Big|_{[b,c]\times[\alpha,\beta]}$ is continuous, then f is in N.

Then N is called an *admitting homotopy relation* (*AHR*).

THEOREM 1. *Let N be an AHR. Let a, b, c, d, α and β be numbers such that $a<b<c<d$ and $\alpha<\beta$, and let Y be a topological space. Let $f : [\alpha,\beta]\times[a,d] \to Y$ be a function such that $f\Big|_{[\alpha,\beta]\times[a,b]}$ and $f\Big|_{[\alpha,\beta]\times[c,d]}$ are in N and $f\Big|_{[\alpha,\beta]\times[b,c]}$ is continuous. Then f is in N.*

Proof. Define $g : [a,d]\times[\alpha,\beta] \to [\alpha,\beta]\times[a,d]$ by if $(x,y)\in[a,d]\times[\alpha,\beta]$, then $g(x,y)=(y,x)$. Then g is a homeomorphism. By Definition 1, parts (iv) and (iii), fg is in N. By Definition 1, part (iii), $f=fgg^{-1}$ is in N.

DEFINITION 2. Let Y be a topological space and let $y_0\in Y$. Let I be the closed unit interval $[0,1]$ with the usual topology and let $C(Y,y_0)$ be the set of all continuous functions $f : I\to Y$ such that $f(0)=y_0=f(1)$. Let N be an AHR. Let $f, g\in C(Y,y_0)$. We say f is *N-homotopic* to g modulo y_0, denoted by $f\simeq_{y_0}g$, provided there is an element $F : I\times I\to Y$ in N such that $F(x,0)=f(x)$, $F(x,1)=g(x)$, and $F(0,t)=y_0=F(1,t)$ for all $x\in I$, $t\in I$.

THEOREM 2. *Let N be an AHR. The relation $\simeq_{y_0}$ is an equivalence relation on $C(Y,y_0)$.*

Proof. Let $f\in C(Y,y_0)$. Since $f \sim_{y_0} f$ [$\sim_{y_0}$ denotes the usual homotopy relation], and since every continuous function is in N, $f\simeq_{y_0}f$.

Let $f, g\in C(Y,y_0)$ and suppose that $f\sim_{y_0}g$. Then there is an element $F : I\times I\to Y$ in N such that $F(x,0)=f(x)$, $F(x,1)=g(x)$, *and* $F(0,t)=y_0=F(1,t)$ for all $X\in I$, $t\in I$. Define $G : I\times I\to Y$ by $G(x,t)=F(x,1-t)$ for all $(x,t)\in I\times I$. Then $G(x,0)=F(x,1)=g(x)$ and $G(x,1)=F(x,0)=f(x)$ for all $x\in I$ and $G(0,t)=F(0,1-t)=y_0=F(1,1-t)=G(1,t)$ for all $t\in I$. Define $K : I\times I\to I\times I$ by $K(x,t)=(x,1-t)$ for all $(x,t)\in I\times I$. Then K is a homeomorphism and thus FK is in N. But if $(x,t)\in I\times I$, then $G(x,t)=F(x,1-t)=F(K(x,t))=FK(x,t)$ and hence G is in N. Hence, $g\simeq_{y_0}f$.

Let $f, g, h\in C(Y,y_0)$ and suppose that $f\simeq_{y_0}g$ and $g\simeq_{y_0}h$. Then there are functions $F, G : I\times I\to Y$ in N such that $F(x,0)=f(x)$, $F(x,1)=g(x)$, and $F(0,t)=y_0=F(1,t)$ for all $x\in I$, $t\in I$ and $G(x,0)=g(x)$, $G(x,1)=h(x)$, and $G(0,t)=y_0=G(1,t)$ for all $x\in I$, $t\in I$, Define $H : I\times I\to Y$ by

$$H(x,t)=\begin{cases} F(x,4t) & \text{if } x\in I,\ 0\le t\le 1/4 \\ g(x) & \text{if } x\in I,\ 1/4\le t\le 3/4. \\ G(x,4t-3) & \text{if } x\in I,\ 3/4\le t\le 1 \end{cases}$$

Define $\alpha : I\times[0,1/4]\rightarrow I\times I$ by $\alpha(x,t)=(x,4t)$ for all $(x,t)\in I\times[0,1/4]$ and define $\beta : I\times[3/4,1]\rightarrow I\times I$ by $\beta(x,t)=(x,4t-3)$ for all $(x,t)\in I\times[3/4,1]$. Then $H|_{I\times[0,1/4]}=F\alpha$ and $H|_{I\times[3/4,1]}=G\beta$. Since F and G are in N and α and β are homeomorphisms, $H|_{I\times[0,1/4]}$ and $H|_{I\times[3/4,1]}$ are in N. Since g is continuous, $H|_{I\times[1/4,3/4]}$ is continuous. By theorem 1, H is in N. Now $H(x,0)=F(x,0)=f(x)$ and $H(x,1)=G(x,1)=h(x)$ for all $x\in I$. Also

$$H(0,t)=\begin{cases} F(0,4t) & \text{if } 0\leq t\leq 1/4,\\ g(0) & \text{if } 1/4\leq t\leq 3/4 = y_0,\\ G(0,4t-3) & \text{if } 3/4\leq t\leq 1\end{cases}$$

for all $t\in I$, and

$$H(1,t)=\begin{cases} F(1,4t) & \text{if } 0\leq t\leq 1/4,\\ g(1) & \text{if } 1/4\leq t\leq 3/4 = y_0,\\ G(1,4t-3) & \text{if } 3/4\leq t\leq 1\end{cases}$$

for all $t\in I$. Therefore, $f\simeq_{y_0}h$. Hence, $\simeq_{y_0}$ is an equivalence relation on $C(Y,y_0)$.

DEFINITION 3. Let $f,g\in C(Y,y_0)$. Then $f \mathbin{\#} g$ is the function in $C(Y,y_0)$ defined by if $x\in I$, then

$$(f \mathbin{\#} g)(x)=\begin{cases} f(4x) & \text{if } 0\leq x\leq 1/4,\\ y_0 & \text{if } 1/4\leq x\leq 3/4,\\ g(4x-3) & \text{if } 3/4\leq x\leq 1.\end{cases}$$

The definition of $\#$ is made this way instead of the usual way as a matter of convenience. Many of the proofs are shorter with this definition. Let N be an AHR. The equivalence relation $\sim_{y_0}$ breaks $C(Y,y_0)$ into disjoint equivalence classes. Let $N(Y,y_0)$ be this set of equivalence classes. If $[f]$, $[g]\in N(Y,y_0)$, then we define $[f]\cdot[g]$ to be $[f \mathbin{\#} g]$.

LEMMA 3.1 *Let N be an AHR. Let $f,g\in C(Y,y_0)$. If $f*g$ is defined in the usual manner*

i. e. *if $x\in X$, then*

$$(f*g)(x)=\begin{cases} f(2x) & \textit{if } 0\leq x\leq 1/2,\\ g(2x-1) & \textit{if } 1/2\leq x\leq 1\end{cases}$$

then $f \mathbin{\#} g\sim_{y_0} f*g$ where by $\sim_{y_0}$ is meant the usual homotopy relation.

Proof. Define $H: I\times I\to Y$ by if $(x,t)\in I\times I$, then

$$H(x,t)=\begin{cases} f\left(\dfrac{4x}{t+1}\right) & \text{if } t\geq 4x-1\\ y_0 & \text{otherwise.}\\ g\left(\dfrac{4x+t-3}{t+1}\right) & \text{if } t\geq -4x+3\end{cases}$$

Then H is the required homotopy.

LEMMA 3.2. *Let N be an AHR. If $[f], [g]\in N(Y, y_0)$, then $[f]\cdot[g]$ is well-defined.*

Proof. Let $f_1, f_2\in[f]$ and $g_1, g_2\in[g]$, Then there are functions F and G in N such that $F: I\times I\to Y$, $G: I\times I\to Y$, $F(x,0)=f_1(x)$, $F(x,1)=f_2(x)$, $F(0,t)=y_0=F(1,t)$, $G(x,0)=g_1(x)$, $G(x,1)=g_2(x)$, and $G(0,t)=y_0=G(1,t)$ for all $x\in I$, $t\in I$. Define a function $H: I\times I\to Y$ by if $(x,t)\in I\times I$, then

$$H(x,t)=\begin{cases} F(4x,t) & \text{if } 0\leq x\leq 1/4\\ y_0 & \text{if } 1/4\leq x\leq 3/4.\\ G(4x-3,t) & \text{if } 3/4\leq x\leq 1\end{cases}$$

Then $H(x,0)=(f_1\# g_1)(x)$, $H(x,1)=(f_2\# g_2)(x)$, and $H(0,t)=y_0=H(1,t)$ for all $x\in I$, $t\in I$. Define $h: [0,1/4]\times I\to I\times I$ by if $(x,t)\in[0,1/4]\times I$, then $h(x,t)=(4x,t)$ and define $k: [3/4,1]\times I\to I\times I$ by if $(x,t)\in[3/4,1]\times I$, then $k(x,t)=(4x-3,t)$. Then $H\big|_{[0,1/4]\times I}=Fh$ and $H\big|_{[3/4,1]\times I}=GK$. Since h and k are homeomorphisms, by definition 1, part (iii), $H\big|_{[0,1/4]\times I}$ and $H\big|_{[3/4,1]\times I}$ are in N. Since $H\big|_{[1/4,\ 3/4]\times I}=y_0$, it is continuous. Hence, by Definition 1, part (iv), H is in N. Thus, $f_1\# g_1\simeq_{y_0} f_2\# g_2$. Thus, $[f_1\# g_1]=[f_2\# g_2]$ and hence $[f]\cdot[g]$ is well-defined.

DEFINITION 4. Let N be an AHR. The *identity element* of $N(Y, y_0)$, denoted by $[e]$, is the equivalence class which contains the function $e: I\to Y$ defined by if $x\in I$, then $e(x)=y_0$.

LEMMA 3.3. *Let N be an AHR. If $[f]\in N(Y, y_0)$,* then $[f]\cdot[e]=[f]$.

Proof. $[f]\cdot[e]=[f\# e]=[f*e]=[f]$.

DEFINITION 5. If $[f]\in N(Y, y_0)$, then $[f]^{-1}$ is the element of $N(Y, y_0)$ containing the function $g: I\to Y$ defined by if $x\in I$, then $g(x)=f(1-x)$.

LEMMA 3.4. *Let N be an AHR.*

If $[f]\in N(Y,y_0)$, *then* $[f]\cdot[f]^{-1}=[e]$.

Proof. Let $[f]\in N(Y,y_0)$. Define $g:I\to Y$ by if $x\in I$, then $g(x)=f(1-x)$. Then $[f]\cdot[f]^{-1}=[f]\cdot[g]=[f\#g]=[f*g]=[e]$.

THEOREM 3. *Let* Y *be a space, let* $y_0\in Y$, *and let* N *be an AHR. Then* $(N(Y,y_0),\ \cdot\ )$ *is a group.*

Proof. Let $[f]$, $[g],[h]\in N(Y,y_0)$. Then $([f]\cdot[g])\cdot[h]=[f\#g]\cdot[h]=[f*g]\cdot[h]=[(f*g)\#h]=[(f*g)*h]=[f*(g*h)]=[f\#(g*h)]=[f]\cdot[g*h]=[f]\cdot[g\#h]=[f]\cdot([g]\cdot[h])$. By Lemmas 3.2, 3.3, and 3.4, $(N(Y,y_0),\ \cdot\ )$ is a group.

From now on, the symbol $N(Y,y_0)$ will mean the set $N(Y,y_0)$ together with the operation $\cdot$ and $N(Y,y_0)$ will be called the first N *group of* Y *with respect to* y_0.

THEOREM 4. *Let* Y *be a space, let* $y_0\in Y$, *and let* N *be an AHR. Then there is an epimorphism* $\lambda:\Pi_1(Y,y_0)\to N(Y,y_0)$.

Proof. Let $[f]\in\Pi_1(Y,y_0)$. Define $\lambda([f])$ to be the equivalence class in $N(Y,y_0)$ which contains f.

Let $[f]\in\Pi_1(Y,y_0)$ and let $f,g\in[f]$. Then $f\sim_{y_0}g$ and thus $f\simeq_{y_0}g$. Thus, λ is well-defined.

Let $M\in N(Y,y_0)$ and let $f\in M$. Then the element $[f]\in\Pi_1(Y,y_0)$ has the property that $\lambda([f])=M$. Therefore, λ is onto.

Let $[f],[g]\in\Pi_1(Y,y_0)$. Then $\lambda([f]\cdot[g])=\lambda([f*g])$. Thus $\lambda([f]\cdot[g])$ is the equivalence class containing $f*g$ in $N(Y,y_0)$. Now $\lambda([f])$ is the equivalence class containng f in $N(Y,y_0)$ and $\lambda([g])$ is the equivalence class containing g in $N(Y,y_0)$. Hence, $\lambda([f])\cdot\lambda([g])$ is the equivalence class in $N(Y,y_0)$ containing $f\#g$. By lemma 3.1, $f*g\sim_{y_0}f\#g$ and therefore, $f*g\simeq_{y_0}f\#g$. Hence, $\lambda([f]\cdot[g])=\lambda([f])\cdot\lambda([g])$, and λ is an epimorphism.

The proofs of the following two theorems are similar to proofs when working with usual homotopy groups and thus are omitted.

THEOREM 5. *If* N *is an AHR. and if* X *and* Y *are spaces,* $x_0\in S$, $y_0\in Y$, *and* H *is a homeomorphism from* X *onto* Y *with* $H(x_0)=y_0$, *then* $N(X,x_0)$ *is isomorphic to* $N(Y,y_0)$.

THEOREM 6. *If* N *is an AHR and* Y *is a pathwise connected space and* y_0, $y_1\in Y$, *then* $N(Y,y_0)$ *is isomorphic to* $N(Y,y_1)$.

3. Questions and comments

In [5] it is shown that connectivity maps [17] form an admitting homotopy relation. It should be noted that here one discovers the reason for the wording in the fourth part of definition of an admitting homotopy relation. For it is not the case that if you have a function which is a connectivity map on the two halves of the square and continuous on the middle line, that the function must be a connectivity map on the entire square.

In [5] it is also shown that if Y is a convex subspace of a real linear topological space, then almost continuous functions [Stallings, 17] form an admitting homotopy relation.

In [4] it is shown that c-continuous functions form something stronger than an admitting homotopy relation. In [4] it is further shown that the first c-continuous group is sometimes non-trival and sometimes different from the usual fundamental group.

Other non-continuous functions which form an admitting homotopy relation are: almost continuous functions [Singal, 16], almost continuous functions [Frolik, 2], almost continuous functions [Husain, 7], almost c-continuous functions [8], H-continuous functions [10], δ-continuous functions [13], somewhat continuous functions [6], weakly continuous functions [9], θ-continuous functions [1], quasi continuous functions [11], and feebly continuous [12].

It is known by the authors that somewhat continuous functions are not fruitful in this context. Namely that the first somewhat continuous group is always trival. In this context very little is known about the other non-continuous functions mentioned in the preceding paragraph.

It would be of interest to know how much if any the concept of admitting homotopy relations helps to distinguish spaces. A place to start would be to pick various examples for which the fundamental group is known and calculate the first N-groups for the various types of known admitting homotopy relations.

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