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A Note on ξ-Zero Regular Spaces

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In this paper, we introduce the concept of ξ-continuity using the ξ-zero set, and generalize the characterization of the ξ-regularness using the ξ-continuity. From these results, we will show that the ξ-product space of ξ-zero regular spaces is ξ-zero regular and ξ-zero regular space has the hereditary property. Also, we will show that ξ-zero regularization is reflective.

Definition 1. Let $\xi=\{E_i : i\epsilon I\}$ be a family of topological spaces. A subset Z of X is said to be a *ξ-zero set* if $Z=\{y\epsilon X : f_i(x)=f_i(y)$ *for* $x\epsilon X\}$ for some continuous map $f_i : X\to E_i\epsilon\xi$.

Definition 2. A map $f : X\to Y$ is said to be *ξ-continuous at a point x* of X if for every ξ-zero set nbd V of $f(x)$, there exists a ξ-zero set nbd U of x such that $f(U)\subset V$. A map $f : X\to Y$ is ξ-continuous on X if it is ξ-continuous at every point of X.

Theorem 1. *If a map $f : X\to Y$ is continuous at $x\epsilon X$, then f is ξ-continuous at x.*

Proof. Sinse f is continuous at $x\epsilon X$, it is sufficient to show that for any ε-zero set V in Y containg $f(x)$, $f^{-1}(V)$ is ξ-zero set in X. Since V is ξ-zero set in Y, there exists $u_i : Y\to E_i$ for some $i\epsilon I$ such that $u_i^{-1}(u_i(f(x)))=V$. On the while, $f^{-1}(V)=f^{-1}(u_i^{-1}(u_i(f(x)))=(u_i\circ f)^{-1}(u_i\circ f)(x)$. Thus $f^{-1}(V)$ is ξ-zero set in X.

Corollary 2. *Identity map $1_X : X\to X$ is ξ-continuous.*

Theorem 3. *If a map $f : X\to Y$ is ξ-continuous at $x\epsilon X$, and a map $g : Y\to Z$ is ξ-continuous at $f(x)$, then $g\circ f : X\to Z$ is ξ-continuous at x.*

Proof. Let W be a ξ-zero set nbd of $g(f(x))$. Then, since g is ξ-continuous at $f(x)$, there exists ξ-zero set nbd V of $f(x)$ such that $g(V)\subset W$. On the other hand, since f is continuous at x, there exists ξ-zero set nbd U of x such that $f(U)\subset V$. Thus $g(f(U))\subset W$, i.e. $(g\circ f)(U)\subset W$. Hence $g\circ f$ is ξ-continuous.

Remark. By the above theorems, the class of topological spaces and ξ-continuous maps form a category.

Definition 3. Let X be a topological space and $(X_\alpha)_{\alpha\epsilon A}$ be a family of topological spaces. A source $(f_\alpha : X\to X_\alpha)_{\alpha\epsilon A}$ is called ξ-initial if it satisfies the following;

1) Each map f_α is ξ-continuous.

2) For any topological space Y and a map $h : Y\to X$, h is ξ-continuous if and only if for each $\alpha\epsilon A$, $f_\alpha\circ h : Y\to X_i$ is ε-continuous.

Theorem 4. *a) If $(f : X\to X_\alpha)_{\alpha\epsilon A}$ and $(g_\beta : X\to X_\beta)_{\beta\epsilon B}$ are sources of ξ-continuous maps, $(f_\alpha)_{\alpha\epsilon A}\subseteq (g_\beta)_{\beta\epsilon B}$, and $(f_\alpha)_{\alpha\epsilon A}$ is ε-initial, then $(g_\beta)_{\beta\epsilon B}$ is ξ-initial.*

b) If $(f_\alpha : X \to X_\alpha)_{\alpha\epsilon A}$ *is ε-initial and* $(g_{\lambda\alpha} : X_\alpha \to Y_{\lambda\alpha})_{\lambda\alpha\epsilon\Lambda\alpha}$ *is ξ-initial for all* $\alpha\epsilon A$, then $(g_{\lambda\alpha}\circ f_\alpha : X \to Y_{\lambda\alpha})_{\lambda\epsilon A,\ \lambda\alpha\epsilon\Lambda\alpha}$ *is ξ-initial.*

c) Let $(f_\alpha : X \to X_\alpha)_{\alpha\epsilon A}$ *be a source of ξ-continuous maps, and for each* $\alpha\epsilon A$, $(g_{\lambda\alpha} : X_\alpha \to Y_{\lambda\alpha})_{\lambda\alpha\epsilon\Lambda\alpha}$ *be a source of ξ-continuous maps. Then, if* $(g_{\lambda\alpha}\circ f_\alpha : X \to Y_{\lambda\alpha})_{\alpha\epsilon A,\ \lambda\alpha\epsilon\Lambda\alpha}$ *is ξ-initial, then* $(f_\alpha : X \to X_\alpha)_{\alpha\epsilon A}$ *is ξ-initial.*

Proof. a) Let $h : Y \to X$ be a map such that $h\circ g_\beta$ is ξ-continuous for all $\beta\epsilon B$. Then, since $(f_\alpha)_{\alpha\epsilon A}\subset(g_\beta)_{\beta\epsilon B}$, $f_\alpha\circ h$ is ξ-continuous for all ${}_{\alpha\epsilon}A$. Since $(f_\alpha)_{\alpha\epsilon A}$ is ξ-initial, h is ξ-continuous.

b) Let $h : Y \to X$ be a map such that $g_{\lambda\alpha}\circ f_\alpha\circ h$ is ξ-continuous for all $\alpha\epsilon A$, $\lambda_\alpha\epsilon\Lambda_\alpha$ since$(g_{\Lambda\alpha})_{\lambda\alpha\epsilon\Lambda\alpha}$ is ξ-initial, $f_\alpha\circ h$ is ξ-continuous for all $\alpha\epsilon A$. Also, since $(f_\alpha)_{\alpha\epsilon A}$ is ξ-initial, h is ξ-continuous.

c) Let's show that if $(f_\alpha\circ h)_{\alpha\epsilon A}$ is ξ-continuous, then h is ξ-continuous. Since $g_{\lambda\alpha}\circ(f_\alpha\circ h)=(g_{\lambda\alpha}\circ f_\alpha)\circ h$ is ξ-continuous, and $(g_{\lambda\alpha}\circ f_\alpha)_{\alpha\epsilon A,\ \lambda\alpha\epsilon\Lambda\alpha}$ is ξ-initial, h is ξ-continuous. Hence $(g_{\lambda\alpha}\circ f_\alpha)_{\alpha\epsilon A,\ \lambda\alpha\epsilon\Lambda\alpha}$ is ξ-initial.

Definition 4. A map $f : X \to Y$ is *ξ-homeomorphism* if f is bijective, f is ξ-continuous, and f^{-1} is ξ-continuous.

Definition 5. Let $\{X_i : i\epsilon I\}$ be any family of topological spaces. The ξ-product topology $\mathcal{T}_\epsilon$ in $\prod_{i\epsilon I} X_i$ is the ξ-initial topology with respect to the source $(Pr_i : \prod_{i\epsilon I} X_i \to X_i)_{i\epsilon I}$. We call $(\prod_{i\epsilon I} X_i,\ \mathcal{T}_\epsilon)$ *ξ-product space.*

Definition 6. A space X is *ξ-zero regular* if X is ξ-homeomorphic with a subspace of ξ-product space $\prod_{E\epsilon\xi'} E$ for some $\xi'\subset\xi$.

Definition 7. A map $f : X \to Y$ is an *ξ-embedding* if f is injective and ξ-initial.

Theorem 5. *A map* $f : X \to Y$ *is an ξ-embedding if and only if* $f : X \to f(X)\subset Y$ *is an ξ-homeomorphism.*

Proof. Since $f : X \to f(X)$ is bijective, there exists $f^{-1} : f(X) \to X$. On the while, since f is ξ-initial and $1_{f(X)}$ is ε-continuous, f^{-1} is ξ-continuous. Thus f is an ξ-homemorphism.

Conversely, let $h : Z \to X$ be a map and $f\circ h : Z \to Y$ be ξ-continuous. Since $f : X \to f(X)$ is an E-homeomorphism, $f^{-1} : f(X) \to X$ is ξ-continuous. Hence $f^{-1}\circ(f\circ h)=h$ is ξ-continuous. Thus f is an ξ-embedding.

Theorem 6. *a)* $(f_\alpha : X \to X_\alpha)_{\alpha\epsilon A}$ *is ξ-initial if and only if* $\prod f_\alpha : X \to \prod X_\alpha$ *is ξ-initial.*

b) $(f_\alpha : X \to X_\alpha)_{\alpha\epsilon A}$ *is ξ-initial mono source if and only if* $\prod f_\alpha : X \to \prod X_\alpha$ *is ξ-embedding.*

Proof. a) By b) and c) of Theorem 4, it is obvious.

b) Let $x \neq y$. Since $(f_\alpha)_{\alpha\epsilon A}$ is mono source, there exists $\alpha\epsilon A$ such that $f_\alpha(x)\neq f_\alpha(y)$. Thus $(\prod f_\alpha)(x)\neq(\prod f_\alpha)(y)$. Hence $\prod f_\alpha$ is an ξ-embedding.

Conversely, let $\prod f_\alpha$ be a mono source. Then for $x\neq y$ in X, $(\prod f_\alpha)(x)\neq(\prod f_\alpha)(y)$. Hence there exists $\alpha\epsilon I$ such that $f_\alpha(x)\neq f_\alpha(y)$. Thus $(f_\alpha : X \to X_\alpha)_{\alpha\epsilon A}$ is a mono source.

Theorem 7. *The following statements are equivalent;*

a) A space X *is ξ-zero regular.*

b) $C(X,\xi)=\bigcup_{E\epsilon\xi}\{f : X \to E;$ *ε-continuous*$\}$ *is ξ-initial mono source.*

c) There exists $\mathcal{F}\subset\bigcup_{E\epsilon\xi} E^{X\alpha}$ *such that* $\mathcal{F}$ *is ξ-initial mono source.*

Proof. a)$\Rightarrow$c). Since X is ξ-zero regular, there exists an ε-embedding $f: X\to \prod_{E\in\xi'\subset\xi} E$. If $x\neq y$, then $f(x)\neq f(y)$. Since f is ξ-embedding, $f(x)\neq f(y)$ for each $x\neq y$ in X. Hence there exists projection Pr_E such that $Pr_E\circ g(x)\neq Pr_E\circ f(y)$. i.e. $\mathcal{F}=(Pr_E\circ f)_{E\in\xi'}$ is ξ-initial mono source.

c)$\Rightarrow$b). Since $\mathcal{F}\subset \prod_{E\in\xi} E^X$ and $\mathcal{F}$ is ξ-initial mono soure, $\mathcal{F}\subset C(X,\xi)$. Hence $C(X,\xi)$ is initial mono source.

b)$\Rightarrow$a). Since $C(X,\xi)$ is ξ-initial mono source, $\prod_{f\in C(X,\xi)} f$ is ξ-embedding by theorem 6, b).

Theorem 8. *Let $\{X_\alpha : \alpha\in A\}$ be a family of ξ-zero regular spaces. If $(f_\alpha : X\to X_\alpha)_{\alpha\in A}$ be ξ-initial mono source, then X is ξ-zero regular.*

Proof. Since for each $\alpha\in A$, X_α is ξ-rero regular, there exists $\mathcal{F}_\alpha\subset \bigcup_{E\in\xi} E^{X_\alpha}$ is ξ-initial mono source. Thus $\bigcup_{\alpha\in A}\{g_{\lambda\alpha}\circ f_\alpha : X\to E,\ g_{\lambda\alpha}\in\mathcal{F}_\alpha\}$ is ξ-initial mono source. Hence X is ξ-zero regular.

Remark. The class of ξ-zero regular spaces and ε-continuous maps form a category which is denoted by *ξ-zero-Reg*.

Corollary 9. *ξ-zero-Reg is ξ-produtive and ξ-hereditary.*

Remark. Let A be a category, and let $ob(A)\supset\xi$. Then *ξ-zero-Reg*$\subset A$.

Lemma 10. *Let X and Y be sets and let $f: X\to Y$ be a maps. Then $ker f=\{(x,y)\mid f(x)=f(y)\}$ is an equivalence relation. Conversely, if R is an equivalence relation, and if $q : X\to X/R$ is a quotient map, then $ker\, q=R$.*

Lemma 11. *Let X, Y and Z be sets. For a surjective map $f: X\to Y$ and a surjective map $g : X\to Z$ there exists a map $h : Y\to Z$ such that $h\circ f=g$ if and only if $ker f\subset ker g$.*

Theorem 12. *Suppose that* $\begin{array}{ccc} X & \xrightarrow{e} & Y \\ f\downarrow & & \downarrow g \\ Z & \xrightarrow[m]{} & P \end{array}$ *commutes and e, m, f, and g are ξ-continuous.*

If e is onto and m is ξ-embedding, then there exists a unique ξ-continuous map $h : Y\to Z$ with $h\circ e=f$ and $m\circ h=g$.

Proof. Let's show that $ker\, e\subset ker\, f$. Let $(x,x')\in ker\, e$, we have $m(f(x))=m(f(x'))$. Since m is injective, it follows that $f(x)=f(x')$, i.e. $(x,x')\in ker f$. Consequently, there exists $h : Y\to Z$ with $h\circ e=f$. Since $m\circ h\circ e=m\circ f$, $m\circ f=g\circ e$, and e is onto, $m\circ h=g$.

Finally, it remains to show that h is ξ-continuous. Since $m\circ h=g$, g *is* ξ-continuous and m is ξ-initial, h is ξ-continuous.

Theorem 13. *Let X be a topological space and let $\varepsilon_X : X\to\varepsilon X$ be an ξ-zero regularization. For every ξ-zero regular space Y, and every map $f\in C(X,\xi)$, there exists a unique ξ-continuous map $\bar{f}$: $\varepsilon X\to Y$ such that $f=\bar{f}\cdot\varepsilon_X$*

Proof. Let $R=\bigcap\{ker f : f\in C(X,\xi)\}$. Then R is an equivalence relation. Let $q : X\to X/R$ be a quotient space. Since for each map $f\in C(X,\xi)$ $R=ker\, q\subset ker\, f$, there exists a unique map $\bar{f} : X/R\to E$ for some $E\in\xi$ such that $\bar{f}\circ g=f$.

Let εX be the space X/R endowed with the ξ-initial topology with respect to $\bigcup_{f\in C(X,\xi)}\{\bar{f} : X/R\to E\in\xi$

and let $\varepsilon_X = q : X \to \varepsilon X$. Then since $\bigcup_{f \in C(X, \varepsilon)} \{\bar{f} : X/R \to E \in \xi\}$ is ξ-initial, and f is ξ-continuous, $\varepsilon_X : X \to \varepsilon X$ is ξ-continuous.

Let's show that $\varepsilon X \in$ ξ*-zero-Reg.* It is enough to show that $\bigcup_{f \in C(X, \varepsilon)} \{\bar{f} : X/R \to E \in \xi\}$ is a mono source. Suppose $[x] \neq [y]$ in $\varepsilon X = X/R$. Then $(x, y) \notin R = \bigcap \{ker f \mid f \in C(X, \varepsilon)\}$. Hence there exists a map $f_o \in C(X, \xi)$ such that $(x, y) \notin ker f_o$. i.e. $f_o(x) \neq f_o(y)$.

Thus $\bar{f}_o([x]) = \bar{f}_o(\varepsilon_X(x)) = f_o(x) \neq f_o(y) = \bar{f}_o([y])$. Hence $\bigcup_{f \in C(X, \varepsilon)} \{\bar{f} : \varepsilon X \to E \in \varepsilon\}$ is ξ-initial mono source. This show that $\varepsilon X \in$ ξ*-zero-Reg.*

Since $Y \in$ ξ*-zero-Reg,* $C(Y, \xi)$ is ξ-initial mono source. For any $u \in C(Y, \xi)$, $u \circ f \in C(X, \xi)$. Therefore there exists a unique ξ-continuous map $\bar{u} : \varepsilon X \to E \in \xi$ such that $\bar{u} \circ \varepsilon_X = u \circ f$.

Let $(x, x') \in ker \varepsilon_X$, then $\varepsilon_X(x) = \varepsilon_X(x')$. Thus $\bar{u} \circ \varepsilon_X(x) = \bar{u} \circ \varepsilon_X(x')$ for any $u \in C(Y, \xi)$, i.e. $u \circ f(x) = u \circ f(x')$ for all $u \in C(X, \xi)$. Since $C(Y, \xi)$ is mono source, $f(x) = f(x')$. Hence $(x, x') \in ker f$.

In all, $ker \varepsilon_X \subset ker f$. Since ε_X is onto, there exists a unique map $\bar{f} : \varepsilon X \to Y$ such that $\bar{f} \circ \varepsilon_X = f$. Since for any $u \in C(Y, \xi)$, $u \circ \bar{f} \circ \varepsilon_X = u \circ f = \bar{u} \circ \varepsilon_X$ and ε_X is onto, $u \circ \bar{f} = \bar{u}$ is ξ-continuous for every $u \in C(Y, \xi)$. Since $C(Y, \xi)$ is ξ-initial, $\bar{f} : \varepsilon X \to Y$ is ξ-continuous.

References

1. Gillman and Jerison, *Rings of Continuous Functions,* D. Van Nostrand Company, Inc. Princeton. 1960.
2. Horst Herrlich, *Categorical Topology* 1971∼1981. Bremen, 1981.