韓國數學敎育學會誌<數學敎育>
1982. 6. Vol. XX, No. 3

Journal of the Korea Society of Mathematical Education
Jun. 1982. Vol. XX, No. 3

On characterizations of σ-quasibarrelled spaces

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I. Introduction

An important property of countably barrelled spaces is that each weakly bounded sequence in the dual space forms an equicontinuous set. This has been used by De Wilde and Houet to introduce the concept of σ-barrelledness. Levin and Saxon have also, independently, introduced this concept, which they call ω-barrelledness to study the hereditary properties of subspaces of such spaces.

A locally convex space E is said to be ***σ-quasibarrelled*** if each $\beta(E', E)$-bounded sequence in E' is equicontinuous.

In 1961, M. Mahowald proved the following result which is a converse of the above mentioned statement:

A locally convex Hausdorff space E is barrelled if for every Banach space F, any closed linear map from E into F is continuous.

In 1976, A. Wilansky has proved the following result:

A locally convex Hausdorff space E is barrelled if for every compact Hausdorff space X any closed graph linear map $T:E\to C(X)$ is continuous.

The purpose of this paper is to give some characterizations of various types of quasibarrelledness which are as follows:

Let E be a Mackey space. Then the following statements are equivalent:

(a) E is β-barrelled

(b) $(E', \beta(E',E))$ is sequentially complete

(c) Every closed graph bounded linear map from E to a separable B_r complete space is continuous

(d) Every closed graph bounded linear map from E to C is continuous.

II. The main theorem

Lemma. *Let E be Mackey space and with $(E', \beta(E',E))$ sequentially complete; let F be separable and B_r complete. Then every linear map with closed graph from E to F is continuous.*

Proof. T is weakly continuous. By Hellinger-Toeplitz, T is continuous since E is Mackey space and the topology on F is smaller than (or equal to) $\tau(F, F')$.

Theorem. *Let E be a Mackey space. Then the following statements are equivalent:*

(a) *E is β-barrelled*

(b) *$(E', \beta(E',E))$ is sequentially complete*

(c) *Every closed graph bounded linear map from E to a separable B_r complete space is continuous.*

(d) *(Every closed graph bounded linear map from E to C is continuous.*

Proof. We will show that: (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(a).

Assume that (a) holds. Let $\{f_n: n\in N\}$ be a Cauchy sequence in $(E', \beta(E', E))$. Then $\{f_n: n\in N\}$ is $\beta(E', E)$-bounded and equicontinuous. Now define $\lim f_n(x)=f_o(x)$ for each $x\in E$. Then f_o is a linear map and $f_o\in E'$ and indeed $\{f_n: n\in N\}$ converges to f_o for the toplogy $\beta(E', E)$. This proves that (a) implies (b).

Assume that (b) holds. Let $T: E\to F$ be a closed graph bounded linear map, where F is separable B_r complete. By Lemma, T is continuous. This completes the proof that (b) implies (c).

Assume that (c) holds. Let C be a Banch space of bounded sequences. Then C is separable and B_r complete. This proves that (c) implies (d) by taking $F=C$.

Assume that (d) holds. Let $\{f_n: n\in N\}$ be $\beta(E', E)$-bounded sequence in E'. Define $T: E\to C$ by $T(x) = \langle f_1(x), f_2(x), \cdots\rangle$. Then clearly T is bounded linear map and also has closed graph. By hypothesis, T is continuous. Then the inverse image of the unit disc in C is a neighborhood of 0 in E. Hence $\{f_n: n\in N\}$ is equicontinuous. This completes the proof that (d) implies (a).

References

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