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A Study on Linear Operations on Stationary Process*

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This paper deals with some properties of the Gaussian stationary processes and the spectral representation.

Our results are motivated by J. Yeh [4]. Unless otherwise stated in this paper most of terminologies and notations come from [1] and [2].

A Gaussian random process $\xi(t)=\xi(\omega, t)$ with values in a probability space Ω, where the parameter t takes integer (discrete) or real values $(-\infty<t<\infty)$, is said to be stationary if its mean is constant

$$a(t)=M\xi(t)\equiv a$$

and the correlation function $B(s,t)$ depends on the difference $(s-t)$ only:

$$B(s,t)=M[\xi(s)-a][\xi(t)-a]=B(s-t). \quad (1)$$

The function $B(t)$ in (1) is said to be a correlation function of the stationary process $\xi(t)$; it can be expressed as

$$B(t)=\int e^{i\lambda t}F(d\lambda), \quad (2)$$

where $F(d\lambda)$ is called the spectral measure of the stationary process $\xi(t)$. In (2) the integration is over $-\pi\leqslant\lambda\leqslant\pi$ in the case of discrete time t and over $-\infty<\lambda<\infty$ in the case of continuous time t.

The stationary process $\xi(t)$ permits a spectral representation of the form

$$\xi(t)=\int e^{i\lambda t}\Phi(d\lambda), \quad (3)$$

where $\Phi(d\lambda)$ is called the stochastic spectral measure such that

$$M\Phi(\Delta_1)\overline{\Phi(\Delta_2)}=F(\Delta_1\cap\Delta_2).$$

Each variable η from the closed linear hull $H(T)$ of the values $\xi(t)$, $t\in T$, permits a spectral representaion of the form

$$\eta=\int \varphi(\lambda)\Phi(d\lambda), \quad (4)$$

where $\varphi(\lambda)$ is the function from the space $L_T(F)$, the real linear hull of the functions $e^{i\lambda t}$ of λ, $t\in T$, closed with respect to the scalar product

$$\langle\varphi_1, \varphi_2\rangle_F=\int\varphi_1(\lambda)\overline{\varphi_2(\lambda)}F(d\lambda). \quad (5)$$

The stochastic integral given by (4) is defined for any function $\varphi\in L_T(F)$ and yields $\eta\in H(T)$. The correspondence $\eta \leftrightarrow\varphi(\lambda)$ is a unitary isomorphism of the Hilbert spaces $H(T)$ and $L_T(F)$:

$$\langle\eta_1, \eta_2\rangle=\langle\varphi_1, \varphi_2\rangle_F. \quad (6)$$

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In the case where the parameter t is continuous and the set T is a finite interval, we can define the space $L_T(F)$ as the closure of the subspace L^0 of all functions of the form

$$\varphi(\lambda)=\int_T e^{i\lambda t}u(t)\,dt \tag{7}$$

where the $u=u(t)$ are infinitely differentiable functions vanishing outside of the interval T. Since the functions $\varphi(\lambda)$ decrease faster than $|\lambda|^{-n}$ as $\lambda\to\infty$, the scalar product (5) can be defined on the subspace L^0 with the help of finite spectral measure as well as any σ-finite measure $G(d\lambda)$ satisfying the condition

$$\int(1+\lambda^2)^{-n}G(d\lambda)<\infty$$

for some integer n.

Let us set

$$\langle\varphi_1,\varphi_2\rangle_G=\int\varphi_1(\lambda)\overline{\varphi_2(\lambda)}G(d\lambda) \tag{8}$$

and define the complete Hilbert space $L_T(G)$ to be closure of all functions of the form (7) by the scalar product given by (8). Let $L_T(G)$ be a Hilbert space of the type indicated. (4) prescribes the random functional $\eta=\eta(\varphi)$ defined on the everywhere dense subspace of functions $L_T(G)\cap L_T(F)$.

Suppose $\eta=\eta(\varphi)$ is a random element from the conjugate space of $L_T(G)$, i.e.,

$$\eta(\varphi)=\langle\varphi,\eta\rangle_G, \tag{9}$$

where $\eta=\eta(\lambda)$ is a Gaussian function with trajectories in the Hilbert space $L_T(G)$. The correlation operator B can then be found from the relations

$$\langle B\varphi_1,\varphi_2\rangle_G=M\eta(\varphi_1)\eta(\varphi_2)=\langle\varphi_1,\varphi_2\rangle_F=\langle A\varphi_1,A\varphi_2\rangle_F=\langle A^*A\varphi_1,\varphi_2\rangle_G,$$

where A is the operator on the Hilbert space $L_T(G)$ into the Hilbert space $L_T(F)$ determined by the equality

$$A\varphi(\lambda)=\varphi(\lambda) \text{ and } \varphi\in L_T(G)\cap L_T(F), \tag{10}$$

where A^* is its adjoint on $L_T(F)$ into $L_T(G)$.

B will be a nuclear operator like any correlation operator. We note that for the finite measure $G(d\lambda)$, (9) is equivalent to a spectral representation of the initial stationary process $\xi(t)$, $t\in T$:

$$\xi(t)=\int e^{-i\lambda t}\eta(\lambda)G(d\lambda),\quad t\in T. \tag{11}$$

In fact, the functions $\varphi(\lambda)=e^{i\lambda t}$ are complete in the Hilbert space $L_T(G)$ and, from (11), $\eta(e^{i\lambda t})=\langle e^{i\lambda t},\eta\rangle_G$, $t\in T$, extends to the whole space $L_T(G)$, the closed linear hull of functions of the type $\varphi(\lambda)=e^{i\lambda t}$.

Theorem 1. *A random process $\xi(t)$, $t\in T$, is a random element of a Hilbert space X if and only if the product $B=A^*A$ is a nuclear on a Hilbert space $L_T(G)$ where the operator A is defined by* (10).

Let $\xi=\xi(t)$ be a Gaussian random function of the parameter $t\in T$ with values $\xi(t)=\xi(\omega,t)$, $\omega\in\Omega$, on a probability space $(\Omega,\mathfrak{A},P)$. We assume that the σ-algebra $\mathfrak{A}$ is generated by $\xi(t)=\xi(\omega,t)$ on Ω as the parameter t runs through the set T; in particular, then, the probability measure P on the σ-algebra $\mathfrak{A}=\mathfrak{A}_\xi$ is Gaussian.

Let P_1 be another Gaussian measure on the σ-algebra $\mathfrak{A}$. It is said to be absolutely continuous with respect to P if $P_1(A)=0$ for $P(A)=0$, $A\in\mathfrak{A}$. It is known that the absolutely continuous measure P_1 is representable as

$$P_1(A)=\int_A p(\omega)P(d\omega),\ \ A\in\mathfrak{A}, \tag{12}$$

where $p(\omega)$ is nonnegative definite function on Ω called a density and designated $p(\omega)=P_1(d\omega)/P(d\omega)$. Measures P_1 and P are said to be *equivalent* if they are mutually absolutely continuous. The measures P_1 and P are said to be orthogonal if there exist nonover-lapping sets A and $A_1\in\mathfrak{A}$ for which

$$P(A)=1,\ \ P(A_1)=0,$$

and

$$P_1(A)=0,\ \ P_1(A_1)=1. \tag{13}$$

The absolute continuity implies in this case that for any $\varepsilon>0$ there is $\delta>0$ such that

$$P_1(A)\leqslant\varepsilon \text{ for } P(A)\leqslant\delta \tag{14}$$

for all $A\in\mathfrak{A}$.

We consider now the Gaussian measures P and P_1 with identical means equal to zero.

Let us define the operator A on the Hilbert space $L_T(F)$ into the Hilbert space $L_T(F_1)$ by

$$A\varphi(\lambda)=\varphi(\lambda) \tag{15}$$

for all $\varphi(\lambda)\in L_T^0$. This condition is equivalent to the fact that the operator A is bounded and has a bounded inverse. This can be expressed as

$$A^*A\asymp^* E, \tag{16}$$

where A^* is the adjoint operator of A, and E is the identity operator; (16) implies that

$$\|(A^*A)\varphi\|_F\asymp\|\varphi\|_F,\ \ \varphi\in L_T(F).$$

Note that

$$\langle A^*A\varphi,\psi\rangle_F=\langle A\varphi,\ A\psi\rangle_{F_1}=\langle\varphi,\psi\rangle_{F_1} \tag{17}$$

for any φ,ψ.

We consider the difference

$$\Delta=E-A^*A. \tag{18}$$

Lemma 1. *If the operator Δ is completely continuous,* (16) *as well as* $\|\varphi\|_F\asymp\|\varphi\|_{F_1}$, $\varphi\in L_T^0$ *will be satisfied if and only if the operator Δ has no eigenvalue equal to one.*

Proof. It is obvious that the condition given by (16) is equivalent to the fact that the operator A^*A is bounded and has a bounded inverse $(A^*A)^{-1}$. Further, since the operator A^*A is positive, the difference $\Delta=E-A^*A$ is such that

$$\delta=\sup_{\|\varphi\|_F=1}\langle\Delta\varphi,\varphi\rangle\leqslant 1.$$

We have

$$\langle\varphi,\varphi\rangle_F-\langle A^*A\varphi,\varphi\rangle_F\leqslant\delta\langle\varphi,\varphi\rangle_F$$

and

$$\langle A^*A\varphi,\varphi\rangle_E\geqslant(1-\delta)\langle\varphi,\varphi\rangle_F.$$

Therefore, the bounded operator $(A^*A)^{-1}$ exists for $\delta\neq 1$. On the other hand, if 1 is the eigenvalue of the operator $E-A^*A$, 0 will be the eigenvalue of the operator A^*A, and therefore the inverse operator $(A^*A)^{-1}$ does not exist.

Theorem 2. *Under the condition given by* $\|\varphi\|_F\asymp\|\varphi\|_{F_1}$ the *Gaussian measures P and P_1 are*

* The relationship $\alpha\asymp\beta$ for variables α and β implies that $0<c_1\leqslant\alpha/\beta\leqslant c_2<\infty$ for some constant c_1 and c_2.

equivalent if and only if $\Delta=E-A^*A$ is Hilbert-Schmidt.

Proof. Let us consider the spectral representation of the bounded symmetric operator Δ:

$$\Delta=\int \mu E(d\mu),$$

where $E(d\mu)$ is the spectral family of projection operators (unitary decomposition). It is seen that

$$A^*A=\int (1-\mu)E(d\mu).$$

We assume that the spectrum of the operator Δ is not purely discrete. Then outside of a neighborhood $(-\varepsilon, \varepsilon)$ there is an infinite number of spectral points, and therefore an infinite number of nonoverlapping intervals $[\mu_k, \mu_{k+1}]$, $k=1, 2, \cdots$, such that all invariant orthogonal subspaces of the form

$$E[\mu_k, \mu_{k+1}]L_T(F)$$

are different from zero. Let us choose an element φ_k, $\|\varphi_k\|_F=1$, from each subspace mentioned above for which

$$\langle A^*A\varphi_k, \varphi_j\rangle_F=\langle\varphi_k, \varphi_j\rangle_{F_1}=\begin{cases}\sigma_k^2 & \text{for } j=k\\ 0 & \text{for } j\neq k,\end{cases}$$

and

$$\mu_k\leq 1-\sigma_k^2\leq\mu_{k+1}, \quad (1-\sigma_k^2)^2\geq\varepsilon^2.$$

The entropy distance r_n between the Gaussian measures P and P_1 on the σ-algebra $\mathfrak{U}_n$ generated by the variables $\eta_k=\eta(\varphi_k)$, $k=1, \cdots, n$ is such that

$$r_n \asymp \sum_{k=1}^{n}(1-\sigma_k^2)^2\geq\varepsilon^2 n.$$

It is seen that $r_n\to\infty$ as $n\to\infty$, Gaussian measures P and P_1 are orthogonal on the σ-algebra $\mathfrak{U}=\lim\limits_{n\to\infty}\mathfrak{U}_n$ Thus, the spectrum of the operator Δ is purely discrete for the equivalent measures P and P_1. If φ_1, $\varphi_2, \cdots$, is a complete orthonormal system of functions with eigenvalues $\mu_1, \mu_2, \cdots$, the condition

$$\sum_k \mu_k^2<\infty \tag{19}$$

is equivalent to the fact that

$$\lim_{n\to\infty} r_n \asymp \sum_{k=1}^{\infty}(1-\sigma_k^2)^2<\infty,$$

where $\sigma_k^2=1-\mu_k$, $k=1, 2, \cdots$, is the complete system of eigenvalues of the operator A^*A and r_n is the entropy distance between P and P_1 on the σ-algebra $\mathfrak{U}_n$ generated by the variables $\eta_k=\eta(\varphi_k)$, $k=1, \cdots, n$. Therefore, (19) is necessary and sufficient condition for the Gaussian measures P and P_1 to be equivalent, as was to be proved.

Let us note that (19) can be written as

$$\sum_{k,j}\langle\Delta\varphi_k, \varphi_j\rangle_F^2<\infty$$

and that for any orthonormal system $\psi_1, \psi_2, \cdots\in L_T(F)$

$$\sum_{k,j}\langle\Delta\psi_k, \psi_j\rangle_F^2=\sum_k[\sum_j\langle\Delta\psi_k, \psi_j\rangle_F]^2=\sum_k\|\Delta\psi_k\|_F^2=\sum_k[\sum_j\langle\Delta\psi_k, \varphi_j\rangle_F^2]$$

$$=\sum_j[\sum_k\langle\psi_k, \Delta\varphi_j\rangle_F^2]\leq\sum_j\|\Delta\varphi_j\|_F^2=\sum_{k,j}\langle\Delta\varphi_k, \varphi_j\rangle_F^2.$$

In the above relations the pertinent inequalities become equalities. It is readily seen that the operator Δ is a Hilbert-Schmidt operator if and only if

$$\sum_{k,j}\langle \Delta\varphi_k, \varphi_j\rangle^2_F<\infty \tag{20}$$

for any complete orthonormal system $\varphi_1, \varphi_2, \cdots$. (20) can immediately be expressed in terms of corelation functionals of the distributions P and P_1, since

$$\langle \Delta\varphi, \psi\rangle_F=\langle\varphi, \psi\rangle_F-\langle A^*A\varphi, \psi\rangle_F=\langle\varphi, \psi\rangle_F-\langle\varphi, \psi\rangle_{F_1}=B(\varphi, \psi)-B_1(\varphi, \psi)$$

for any $\varphi, \psi\in L_T(F)$. Therefore, under the condition given by $\|\varphi\|_F\asymp\|\varphi\|_{F1}$ a necessary and sufficient condition for equivalence of the Gaussian measures P and P_1 is that for any complete orthonormal system $\varphi_1, \varphi_2, \cdots\in L_T(F)$

$$\sum_{k,j} b(\varphi_k, \varphi_j)^2<\infty \tag{21}$$

where $b(\varphi, \psi)=B(\varphi, \psi)-B_1(\varphi, \psi)$, $\phi, \psi\in L_T(F)$.

Let us consider actually the linear space L^0_{TXT} of functions of the form

$$\varphi(\lambda, \mu)=\sum_{k,j} c_{kj}e^{i(\lambda s_k-\mu t_j)} \tag{22}$$

(where s_k, $t_j\in T$ and where c_{kj} are real coefficients). We define $L_{T\times T}(F\times F)$ as a Hilbert space obtained by means of the closure of $L^0_{TT\times}$ over the scalar product

$$\langle\varphi, \psi\rangle_{F\times F}=\iint\varphi(\lambda, \mu)\overline{\psi(\lambda, \mu)}F(d\lambda)F(d\mu) \tag{23}$$

It is obvious that if $\varphi', \varphi''\in L_T(F)$, then

$$\varphi(\lambda, \mu)=\varphi'(\lambda)\overline{\varphi''(\mu)} \tag{24}$$

enters into the space $L_{T\times T}(F\times F)$. In this case the system of functions of the type given by (24) is complete in $L_{T\times T}(E\times F)$.

$$\begin{aligned}\|\varphi(\lambda, \mu)-\psi(\lambda, \mu)\|^2_{F\times F}&=\iint|\varphi'(\lambda)\overline{\varphi''(\mu)}-\psi'(\lambda)\overline{\psi''(\mu)}|^2F(d\lambda)F(d\mu)\\ &\leq 2\iint[|\varphi''(\mu)|^2|\varphi'(\lambda)-\psi'(\lambda)|^2+|\psi'(\lambda)|^2|\varphi''(\mu)-\psi''(\mu)|^2]F(d\lambda)F(d\mu)\\ &=2[\|\varphi''\|\|^2_F\varphi'-\psi'\|^2_F+\|\psi^{2\prime}\|_F\|\varphi'-\psi''\|^2_F]\end{aligned}$$

for any functions φ', φ'' and ψ', ψ'' of the type mentioned above.

Let $\varphi_1, \varphi_2, \cdots$. be an orthonormal system in a Hilbert space $L_T(F)$. Then, obviously, the functions

$$\varphi_{k,j}(\lambda, \mu)=\varphi_k(\lambda)\overline{\varphi_j(\mu)} \qquad k, j=1, 2, \cdots,$$

comprise a complete orthonormal system in the Hilbert space $L_{T\times T}(F\times F)$. Under the condition given by (20) let

$$b_{k,j}=b(\varphi_k,\varphi_j), \qquad k, j=1, 2, \cdots,$$

and let $\psi_0(\lambda, \mu)\in L_{T\times T}(F\times F)$ be defined as

$$\psi_0(\lambda, \mu)=\sum_{k,j} b_{kj}\varphi_{kj}(\lambda, \mu) \tag{25}$$

(25) yields the decomposition of $\psi_0(\lambda, \mu)$ over the orthonormal system $\varphi_{kj}(\lambda, \mu)$ so that

$$b(\varphi_k, \varphi_j)=\langle\varphi_k\overline{\varphi_j}, \psi_0\rangle_{F\times F}.$$

This relation obviously extends to any linear combinations

$$\varphi'(\lambda)=\sum_k c_k'\varphi_k(\lambda), \quad \varphi''(\mu)=\sum_j c_j''\varphi_j(\mu),$$

so that

$$b(\varphi', \varphi'')=\langle\varphi'\overline{\varphi''}, \psi_0\rangle_{F\times F}. \tag{26}$$

Theorem 3. *The Gaussian measuers P and P_1 (with zero mean values) are equivalent on the σ-algebra $\mathfrak{A}(T)$ if and only if the difference between the correlation functions*

$$b(s,t)=B(s,t)-B_1(s,t)$$

is representable as

$$b(s,t)=\iint e^{-i(\lambda s-\mu t)}\psi(\lambda,\mu)F(d\lambda)F_1(d\mu) \tag{27}$$

for $s,t\in T$, *where the function* $\psi(\lambda,\mu)$ *is such that*

$$\iint|\psi(\lambda,\mu)|^2F(d\lambda)F(d\lambda)F_1(d\mu)<\infty.$$

For the equivalent measures P *and* P_1 *the integral equation given by* (27) *has a solution* $\psi(\lambda,\mu)\in L_{T\times T}(F\times F_1)$. *The density* $p(\omega)=P_1(d\omega)/P(d\omega)$ *of the equivalent measures can be expressed as*

$$p(\omega)=D\ exp\left\{-\frac{1}{2}\iint\psi(\lambda,\mu)\Psi(d\lambda,d\mu)\right\}, \tag{28}$$

where $\psi(\lambda,\mu)$ *is the solution of* (27) *from the space* $L_{T\times T}(F\times F_1)$; D *is a normalizing multiplier.*

Proof. Since the functions $\varphi(\lambda,\mu)=e^{i(\lambda s-\mu t)}$, $s,t\in T$, form a complete system in the Hilbert space $L_{T\times T}(F\times F_1)$, the relation given by (27) is equivalent, for $\psi(\lambda,\mu)\in L_{T\times T}(F\times F_1)$. If (27) holds true for a function $\psi(\lambda,\mu)$ in the Hilbert space of all square-integrable, the projection of $\psi(\lambda,\mu)$ onto the subspace $L_{T\times T}(F\times F_1)$. Let us consider the densities $p_n(\omega)=P_1(d\omega)/P(d\omega)$ on the σ-algebras $\mathfrak{A}_n$, each of which is generated by variables η_k $k=1,\cdots n$.

$$\log p_n(\omega)=M\log p_n-\frac{1}{2}\sum_{k,j=1}c_{kj}[\xi(t_k)\xi(t_j)-B(t_k,t_j)], \tag{29}$$

where $\{c_{kj}\}$ is the difference between the matrices inverse to the correlation matrices $\{B_1(t_k,t_j)\}$ and $\{B(t_k,t_j)\}$. The corresponding variables

$$\eta_k(\omega)=\sum_{k,j=1}^{n}c_{ki}[\xi(t_k)\xi(t_j)-B(t_k,t_j)]$$

appearing in (29) belong to the space $H_2(T)$and are representable

$$\eta_n=\iint\psi_n(\lambda,\mu)\Psi(d\lambda,d\mu), \tag{30}$$

where

$$\psi_n(\lambda,\mu)=\sum_{k,j=1}^{n}c_{kj}e^{i(\lambda t_k-\mu t_j)} \tag{31}$$

It is easy to verify that each function $\psi_n(\lambda,\mu)$ satisfies an equation of the type given by (27),

$$\iint e^{-i(\lambda s-\mu t)}\psi_n(\lambda,\mu)F(d\lambda)F_1(d\mu)=b(s,t), \tag{32}$$

for $s,t=t_1,\cdots,t_n$. In fact, this equality can be rewritten in matrix form

$$\{B(t_k,t_j)\}\ \{c_{kj}\}\ \{B_1(t_k,t_j)\}=\{b(t_k,t_j)\},$$

where

$$\{c_{kj}\}=\{B_1(t_k,t_j)\}^{-1}-\{B(t_k,t_j)^{-1},$$

and it follows immediately that

$$\{B(t_k,t_j)\}\ \{c_{kj}\}=\{B(t_k,t_j)\}\ \{B_1(t_k,t_j)\}^{-1}-E,$$

$$\{B(t_k,t_j)\}\ \{c_{kj}\}\ \{B_1(t_k,t_j)\}=\{B(t_k,t_j)\}-\{B_1(t_k,t_j)\}=\{b(t_k,t_j)\}$$

We can rewrite (32) also as

$$\langle\varphi,\psi_n\rangle_{F\times F_1}=b(s,t),\quad s,t\in T_n,$$

where $T_n=\{t_1,\cdots,t_n\}$ and $\varphi(\lambda,\mu)=e^{i(\lambda s-\mu t)}$, $s,t\in T_n$. It is clear that for $m\leq n$ the function $\psi_m(\lambda,\mu)$ coincides with the projection of the element $\psi_n(\lambda,\mu)\in L_{T_n\times T_m}(F\times F_1)$ onto the subspace $L_{T_n\times T_m}(F\times F_1)$ so that

$$\|\psi_n-\psi_m\|^2_{F\times F_1}=\|\psi_m\|^2_{F\times F_1}-\|^2\psi_n\|_{F\times F_1}\to 0$$

as $m, n\to\infty$, since the sequence $\|\psi_n\|^2{}_{F\times F_1}$, $n=1, 2, \cdots$, turns out to be monotone decreasing and lim $\|\psi_n\|^2$ exists.

It is also seen that since the Hilbert space $L_{T\times T}(F\times F_1)$ coincides with the closure of extending spaces $L_{T_n\times T_n}(F\times F_1)$, $n=1, 2, \cdots$, with each function $\psi_n(\lambda, \mu)$ in (32) being the projection of $\psi(\lambda, \mu)\in L_{T\times T}(F\times F_1)$ by (27), $\lim_{n\to\infty}\psi_n(\lambda, \mu)\in L_{T\times T}(F\times F_1)$ has the property that

$$\psi(\lambda, \mu)=\lim_{n\to\infty}\psi_n(\lambda, \mu).$$

This fact implies that the variables η_n of the type given by (30) appearing in (29) converges in the mean to the variable

$$\eta=\iint\psi(\lambda, \mu)\Psi(d\lambda, d\mu)\in H_2(T).$$

Therefore, the density $p(\omega)=P_1(d\omega)/P(d\omega)$ on the σ-algebra $\mathfrak{A}(T)$ can be determined by the limit relation given by

$$\begin{aligned}\log p(\omega)&=\lim_{n\to\infty}M\log p_n+\lim_{n\to\infty}[\log p_n-M\log p_n]\\&=\lim_{n\to\infty}M\log p_n-\frac{1}{2}\lim_{n\to\infty}\eta_n(\omega)\\&=\lim_{n\to\infty}M\log p_n-\frac{1}{2}\lim_{n\to\infty}\iint\psi_n(\lambda, \mu)\Psi(d\lambda, d\mu)\\&=\log D-\frac{1}{2}\iint\psi(\lambda, \mu)\Psi(d\lambda, d\mu),\end{aligned}$$

which yields (28).

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