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A subcategory of a properly fibred topological category defined by subinitiality and subfinality.

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1. Introduction

The main purpose of this paper is to introduce new subcategories, $\mathrm{INI}_S(\underline{A}:\underline{B})$ and FIN_S $(\underline{A}:\underline{B})$, in terms of subinitiality and subfinality, respectively, to construct some theorems by means of the new subcategories, and to investigate the properties of the theorems.

In section 2, some preexisting definitions and theorems necessary for the main theorems are introduced.

In section 3, new subcategories are introduced, some theorems by means of them are constructed, and the properties of the theorems are investigated.

2. Preliminaries

Definition 2.1. A category $\underline{A}$ is called *concrete* if the following are satisfied:

(1) objects of $\underline{A}$ are pairs (X,ξ) where X is a set, called the underlying set of (X,ξ), and ξ is some $\underline{A}$-structure on X, called the underlying $\underline{A}$-structure of (X,ξ).

(2) morphisms $f:(X,\xi)\to(Y,\eta)$ of $\underline{A}$ are certain maps $f:X\to Y$, subject to the following conditions:

(i) for each object (X,ξ) of $\underline{A}$, the identity map $1_X:(X,\xi)\to(X,\xi)$ is a morphism;

(ii) if $f:(X,\xi)\to(Y,\eta)$ and $g:(Y,\eta)\to(Z,\zeta)$ are morphisms, then so is $g\circ f:(X,\xi)\to(Z,\zeta)$.

Let $\hom((X,\xi),\ (Y,\eta))$ be the set of morphisms with domain (X,ξ) and range (Y,η).

In the following, every category means a concrete category.

Definition 2.2. A category $\underline{B}$ is called a ***subcategory of a category*** $\underline{A}$ provided that the following conditions are satisfied:

(1) The objects of $\underline{B}$ are also objects of $\underline{A}$

(2) For objects (X,ξ) and (Y,η) of $\underline{B}$, $\hom_B((X,\xi),\ (Y,\eta))\subset\hom_A((X,\xi),\ (Y,\eta))$

(3) If $f:(X,\xi)\to(Y,\eta)$ and $g:(Y,\eta)\to(Z,\zeta)$ are morphisms of $\underline{B}$, their composite in $\underline{B}$ equals their composite in $\underline{A}$.

Definition 2.3. A category $\underline{B}$ is called a ***full subcategory*** of $\underline{A}$ provided that for objects (X,ξ) and (Y,η) in $\underline{B}$, $\hom_B((X,\xi),(Y,\eta))=\hom_A((X,\xi),(Y,\eta))$.

Definition 2.4. Let $\underline{B}$ be a subcategory of $\underline{A}$.

$\underline{B}$ is said to be an ***isomorphism-closed subcategory*** of $\underline{A}$ provided that every $\underline{A}$-object that is isomorphic with some $\underline{B}$-object is itself a $\underline{B}$-object.

In the following, every subcategory will be assumed to be full and isomorphism-closed.

Definition 2.5. (1) A *source* in a category $\underline{A}$ is a pair $(X, (f_i)_{i\in I})$, where X is an $\underline{A}$-object and $(f_i: X\to X_i)_{i\in I}$ is a family of $\underline{A}$-morphisms each with domain X. In this case X is called the *domain of the source* and the family $(X_i)_{i\in I}$ is called the *codomain of the source.*

To simplify notation a source $(X, (f_i)_{i\in I})$ is often denoted by (X, f_i).

(2) A source (X, f_i) is called a *mono-source* provided that the f_i can be simultaneously cancelled from the left; i.e., provided that for any pair $Y \overset{r}{\underset{s}{\rightrightarrows}} X$ of morphisms such that $f_i\circ r=f_i\circ s$ for each $i\in I$, it follows that $r=s$.

Dual notions: sink in $\underline{A}$; (f_i, X); codomain of a sink; domain of a sink.

Definition 2.6. Let $\underline{A}$ be a category and $((Y_i, \eta_i)_{i\in I})$ a family of objects in $\underline{A}$ indexed by a class I, and let X be a set and $(f_i: X\to Y_i)_{i\in I}$ a source of maps indexed by I. An $\underline{A}$-structure ξ on X is called *initial* with respect to $(X, (f_i)_{i\in I}, (Y_i, \eta_i)_{i\in I})$ if the following conditions are satisfied:

(1) for each $i\in I$, $f_i: (X, \xi)\to (Y_i, \eta_i)$ is an $\underline{A}$-morphism,

(2) if (Z, ζ) is an $\underline{A}$-object and $g: Z\to X$ is a map such that for each $i\in I$, the map $f_i\circ g: (Z, \zeta)\to (Y_i, \eta_i)$ is an $\underline{A}$-morphism, then $g: (Z, \zeta)\to (X, \xi)$ is an $\underline{A}$-morphism.

In this case, the source $(f_i: (X, \xi)\to (Y_i, \eta_i))_{i\in I}$ is also called initial.

Dually we define the final structures and final sinks.

Definition 2.7. An $\underline{A}$-product of a family $(A_i)_{i\in I}$ of $\underline{A}$-objects is a pair $(\prod (A_i)_{i\in I}, (\pi_i)_{i\in I})$ satisfying the following properties:

(1) $\prod (A_i)_{i\in I}$ is an $\underline{A}$-object.

(2) for each $j\in I$, $\pi_j: \prod (A_i)_{i\in I}\to A_j$ is an $\underline{A}$-morphism (called the projection from $\prod (A_i)_{i\in I}$ to A_j),

(3) for each pair $(C, (f_i)_{i\in I})$, (where C is a $\underline{A}$-object and for each $j\in I$, $f_j: C\to A_j$) there exists a uniqe $\underline{A}$-morphism (usually denoted by) $\langle f_i\rangle: C\to \prod (A_i)_{i\in I}$ such that for each $j\in I$, $f_j=\pi_j\circ\langle f_i\rangle$.

Lemma 2.8. (1) *$(f_i: X\to X_i)_{i\in I}$ and $(g_j: X\to X_j)_{j\in J}$ are sources in a category $\underline{A}$, and $(f_i)_{i\in I}\subset (g_j)_{j\in J}$ and $(f_i)_{i\in I}$ is initial, then $(g_j)_{j\in J}$ is initial.*

(2) *If $(f_i: X\to X_i)_{i\in I}$ is initial and $(g_{\lambda_i}: X_i\to Y_{\lambda_i})_{\lambda_i\in\Lambda_i}$ is initial for all $i\in I$, then $(g_{\lambda_i}\circ f_i: X\to Y_{\lambda_i})_{i\in I, \lambda_i\in\Lambda_i}$ is initial.*

(3) *If $(f_i: X\to X_i)_{i\in I}$ is a source in $\underline{A}$, for each $i\in I$ $(g_{\lambda_i}: X_i\to Y_{\lambda_i})_{\lambda_i\in\Lambda_i}$ is a source in $\underline{A}$ and $(g_{\lambda_i}\circ f_i: X\to Y_{\lambda_i})_{i\in I, \lambda_i\in\Lambda_i}$ is initial, then $(f_i: X\to X_i)_{i\in I}$ is initial.*

Examples 2.9. (1) (X, f) is a mono-source if and only if f is a monomorphism.

(2) Each product $(\prod X_i, \pi_i)$ is a mono-source.

Definition 2.10. An $\underline{A}$-coproduct of a family $(A_i)_{i\in I}$ of $\underline{A}$-objects is a pair $((u_i)_{i\in I}, \coprod (A_i)_{i\in I})$ satisfying the following properties:

(1) $\coprod (A_i)_{i\in I}$ is an $\underline{A}$-object.

(2) for each $j\in I$, $u_j: A_j\to \coprod (A_i)_{i\in I}$ is an $\underline{A}$-morphism.

(3) for each pair $((f_i)_{i\in I}, A)$, there exists a unique $\underline{A}$-morphism $[f_i]: \coprod (A_i)_{i\in I}\to A$ such that for each $j\in I$, $[f_i]\circ u_j=f_j$

Definition 2.11. A category $\underline{A}$ is said to be *topological* if for each set X, for any family $((Y_i, \xi_i))_{i\in I}$ of $\underline{A}$-objects, and for any family $(f_i: X\to Y_i)_{i\in I}$ of maps, there exists an $\underline{A}$-structure on X

which is initial with respect to $(X, (f_i)_{i\in I}, ((Y_i, \xi_i))_{i\in I})$.

Dually we define cotopological categories.

Definition 2.12. A morphism $f:(X,\xi)\to(Y,\eta)$ in a category is called

(1) an *embedding* if $f:X\to Y$ is $1-1$ and initial,

(2) a *quotient map* if $f:X\to Y$ is onto and final.

Definition 2.13. Let A be a category. If (X,ξ) and (Y,η) are $\underline{A}$-objects, then (X,ξ) is called

(1) a *subspace* of (Y,η) if there is an embedding $f:(X,\xi)\to(Y,\eta)$,

(2) a *quotient space* of (Y,η) if there is a quotient map $q:(Y,\eta)\to(X,\xi)$.

Definition 2.14. Let $\underline{A}$ be a category.

(1) The *$\underline{A}$-fibre* of a set X is the class of all $\underline{A}$-structures on X.

(2) A is called *properly fibred* if it satisfies the following conditions:

(i) for each set X, the $\underline{A}$-fibre of X is a set.

(ii) for each one-element set X, the $\underline{A}$-fibre of X has precisely one element.

(iii) if ξ and η are $\underline{A}$-structures on X such that $1_X:(X,\xi)\to(X,\eta)$ and $1_X:(X,\eta)\to(X,\xi)$ are morphisms, then $\xi=\eta$.

Definition 2.15. Let $\underline{C}$ be a category and $\underline{A}$ a subcategory of $\underline{C}$.

For any $X\in\underline{C}$, a $\underline{C}$-morphism $f:X\to A$ is called the *$\underline{A}$-reflection* of X if $A\in\underline{A}$ and for any $A'\in\underline{A}$ and a $\underline{C}$-morphism $g:X\to A'$, there exist a unique $\underline{A}$-morphism $\bar{f}:A\to A'$ with $\bar{f}\circ f=g$.

If every object of $\underline{C}$ has the $\underline{A}$-reflection, then $\underline{A}$ is called a *reflective subcategory* of $\underline{C}$.

Dual notions; coreflection; coreflectives subcategory.

The following propositions are well-known.

Proposition 2.16. *If $\underline{A}$ is a properly fibred topological category and $\underline{B}$ is a full isomorphism-closed subcategory of $\underline{A}$, then the following are equivalent:*

(1) *$\underline{B}$ is epireflective in $\underline{A}$.*

(2) *$\underline{B}$ is closed under the formation of initial mono-sources.*

(3) *$\underline{B}$ is closed under the formation of products and subspaces in $\underline{A}$.*

Proposition 2.17. *Let $\underline{B}$ be full isomorphism-closed subcategory of a properly fibred topological category $\underline{A}$. If $\underline{B}$ contains at least one object with non-empty underlying set, then the following are equivalent:*

(1) *$\underline{B}$ is coreflective in $\underline{A}$.*

(2) *$\underline{B}$ is bicoreflective in $\underline{A}$.*

(3) *$\underline{B}$ is closed under the formation of final sinks.*

(4) *$\underline{B}$ is closed under the formation of coproducts and quotient spaces in $\underline{A}$.*

3. Main theorems

Definition 3.1. Let $\underline{A}$ and $\underline{B}$ be subcategories of a properly fibred topological category $\underline{C}$.

(1) An object X of $\underline{C}$ is said to be *subinitially defined from $\underline{A}$ to $\underline{B}$* if X^* endowed with the initial structure on X with respect to $\bigcup_{A\in A'}$ hom (X,A) for some $\underline{A'}\subset\underline{A}$ belongs to $\underline{B}$.

(2) The subcategory determined by subinitially defined objects from $\underline{A}$ to $\underline{B}$ is called *subinitially defined subcategory from $\underline{A}$ to $\underline{B}$* and denoted by $\mathrm{INI}_S(\underline{A}:\underline{B})$

In the following theorems, let $\underline{A}$ and $\underline{B}$ be subcategories of a properly fibred topologicall category $\underline{C}$.

Theorem 3.2. *If* $\underline{B}$ *is productive, then* $INI_S(\underline{A}:\underline{B})$ *is productive.*

Proof. Let $(X_i)_{i\in I}$ be a family of objects in $\mathrm{INI}_S(\underline{A}:\underline{B})$

For each $i\in I$, let X_i^* be the object endowed with a initial structure on X_i with respect to $\bigcup_{A\in\underline{A}'}\hom(X_i, A)$ for some $\underline{A}'\subset\underline{A}$. $X_i^*\in\underline{B}$ for each $i\in I$. Since $\underline{B}$ is productive, $\prod X_i^*\in\underline{B}$.

Let $P_{r_i}:\prod X_i^*\to X_i^*$ be the projection for all $i\in I$. For each $A\in\underline{A}'$ and for any $f\in\bigcup_{A\in\underline{A}'}\hom(X_i, A)$, the composite map $f\circ p_{r_i}:\prod X_i\to A$ is a morphism in $\underline{C}$. Since $(f\circ p_{r_i})_{f\in\bigcup_{A\in\underline{A}'}\hom(X_i,A),\, i\in I}\subset\bigcup_{A\in\underline{A}'}\hom(\prod X_i, A)$ and $(f\circ p_{r_i})_{f\in\bigcup_{A\in\underline{A}'}\hom(X_i,A),\, i\in I}$ is initial, $\bigcup_{A\in\underline{A}'}\hom(\prod X_i, A)$ is initial, i.e. $\prod X_i^*$ is the object endowed with the initial structure on $\prod X_i$ with respect to $\bigcup_{A\in\underline{A}'}\hom(\prod X_i, A)$. Hence $\prod(X_i)_{i\in I}\in\mathrm{INI}_S(\underline{A}:\underline{B})$

Theorem 3.3. *If* $\underline{B}$ *is hereditary, then* $INI_S(\underline{A}:\underline{B})$ *is hereditary.*

Proof. Let X^* be the object in $\underline{C}$ endowed with a initial structure on X with respect to $\bigcup_{A\in\underline{A}'}\hom(X, A)$ for some $\underline{A}'\subset\underline{A}$, and let Y^* be a subspace of X^*. Then $X^*\in\underline{B}$. Since $\underline{B}$ is hereditary, $Y^*\in\underline{B}$. Let $j:Y^*\to X^*$ be an embedding and $f\in\bigcup_{A\in\underline{A}'}\hom(X^*, A)$.

Since $(f\circ j)_{f\in\bigcup_{A\in\underline{A}'}\hom(X,A)}\subset\bigcup_{A\in\underline{A}'}\hom(Y, A)$ and $(f\circ j)_{f\in\bigcup_{A\in\underline{A}'}\hom(X,A)}$ is initial, $\bigcup_{A\in\underline{A}'}\hom(Y, A)$ is initial. Thus Y^* is the object of $\underline{C}$ endowed with the initial structure on Y with respect to $\bigcup_{A\in\underline{A}'}\hom(Y, A)$. Hence $Y\in\mathrm{INI}_S(\underline{A}:\underline{B})$

Corollary 3.4. *If* $\underline{B}$ *is epireflective in* $\underline{C}$, *then* $INI_S(\underline{A}:\underline{B})$ *is epireflective in* $\underline{C}$.

Theorem 3.5. *If* $\underline{B}$ *is closed under the formation of epimorphisms, then so is* $INI_S(\underline{A}:\underline{B})$.

Proof. Let $X\in\mathrm{INI}_S(\underline{A}:\underline{B})$, and let $f:X\to Y$ be an epimorphism.

Let X^* be the space endowed with a initial structure on X with respect to $\bigcup_{A\in\underline{A}'}\hom(X, A)$ for some $\underline{A}'\subset\underline{A}$.

For any $g\in\hom(Y, A)$, $gof\in\hom(X, A)$ and hence $g\circ f\in\hom(X^*, A)$. Since for any $A\in\underline{A}'$, $X^*\xrightarrow{g\circ f}A=X^*\xrightarrow{f}Y^*\xrightarrow{g}A$, and $\bigcup_{A\in\underline{A}'}\hom(Y^*, A)$ is initial, $f:X^*\to Y^*$ is epimorphism in $\underline{B}$. Hence $Y^*\in\underline{B}$. In all, $Y\in\mathrm{INI}_S(\underline{A}:\underline{B})$

Definition 3.6. Let $\underline{A}$ and $\underline{B}$ be subcategories of a properly fibred topological category $\underline{C}$.

(1) An object X of $\underline{C}$ is said to be ***subfinally defined from*** $\underline{A}$ ***to*** $\underline{B}$ if X_* endowed with the final structure on X with respect to $\bigcup_{A\in\underline{A}'}\hom(A, X)$ for some $\underline{A}'\subset\underline{A}$ belongs to $\underline{B}$.

(2) The subcategory determined by subfinally defined objects from $\underline{A}$ to $\underline{B}$ and denoted by $\mathrm{FIN}_S(\underline{A}:\underline{B})$

In the following theorems, let $\underline{A}$ and $\underline{B}$ be subcategories of a properly fibred topological category $\underline{C}$.

Theorem 3.7. *If* $\underline{B}$ *is coproductive, then* $FIN_S(\underline{A}:\underline{B})$ *is coproductive.*

Proof. Let $(X_i)_{i\in I}$ be a family of objects in $\mathrm{FIN}_S(\underline{A}:\underline{B})$.

For each $i\in I$, let X_{i*} be the object endowed with a final structure on X_i with respect to $\bigcup_{A\in\underline{A}'}\hom(A, X_i)$ for some $\underline{A}'\subset\underline{A}$. $X_{i*}\in B$ for each $i\in I$. Since $\underline{B}$ is coproductive, $\amalg X_{i*}\in B$. Let $u_i: X_i\to\amalg X_{i*}$ be the inclusion for all $i\in I$. For each $A\in\underline{A}'$ and for any $f\in\bigcup_{A\in\underline{A}'}\hom(A, X_i)$, the composite map $u_i\circ f: A\to\amalg X_i$ is a morphism in $\underline{C}$.

Since $(u_i\circ f)_{f\in\bigcup_{A\in\underline{A}'}\hom(A,X),i\in I}\subset\bigcup_{A\in\underline{A}'}\hom(A,\amalg X_i)$ and $(u_i\circ f)_{f\in\bigcup_{A\in\underline{A}'}\hom(A,\amalg X_i),i\in I}$ is final, i.e. $\amalg X_{i*}$ is the object endowed with the final structure on $\amalg X_i$ with respect to $\bigcup_{A\in\underline{A}'}\hom(A, \amalg X_i)$. Hence $\amalg(X_i)_{i\in I}\in\mathrm{FIN}_S(\underline{A}:\underline{B})$.

Theorem 3.8. *If $\underline{B}$ is cohereditary, then $FIN_S(\underline{A}:\underline{B})$ is cohereditary.*

Proof. Let X_* be the object in $\underline{C}$ endowed with a final structure on X with respect to $\bigcup_{A\in\underline{A}'}\hom(A, X)$ for some $\underline{A}'\subset\underline{A}$, and let Y_* be a quotient space of X_*. $X_*\in\underline{B}$. Since $\underline{B}$ is cohereditary, $Y_*\in\underline{B}$. Let $g: X_*\to Y_*$ be a quotient and $f\in\bigcup_{A\in\underline{A}'}\hom(A, X_*)$. Since $(g\circ f)_{f\in\bigcup_{A\in\underline{A}'}\hom(A,X)}\subset\bigcup_{A\in\underline{A}'}\hom(A, Y)$ and $(g\circ f)_{f\in\bigcup_{A\in\underline{A}'}\hom(A,X)}$ is final, $\bigcup_{A\in\underline{A}'}\hom(A, Y)$ is final.

Thus Y_* is the object of $\underline{C}$ endowed with the final structure on Y with respect to $\bigcup_{A\in\underline{A}'}\hom(A, Y)$. Hence $Y\in\mathrm{FIN}_S(\underline{A}:\underline{B})$.

Corollary 3.9. *If $\underline{B}$ is coreflective in $\underline{C}$, then $FIN_S(\underline{A}:\underline{B})$ is coreflective in $\underline{C}$.*

Theorem 3.10. *If $\underline{B}$ is closed under the formation of monomorphisms, then so is $FIN_S(\underline{A}:\underline{B})$.*

Proof. Let $X\in\mathrm{FIN}_S(\underline{A}:\underline{B})$, and let $g: Y\to X$ be monorphism. Let X_* be the space endowed with a final structure on X with respect to $\bigcup_{A\in\underline{A}'}\hom(A,X)$ for some $\underline{A}'\subset\underline{A}$. For any $h\in\hom(A, X)$, $g\circ h\in\hom(A, X)$ and hence $g\circ h\in\hom(A, X_*)$. Since for any $A\in\underline{A}'$, $A\xrightarrow{g\circ h}X_*=A\xrightarrow{h}Y_*\xrightarrow{g}X_*$, and $\bigcup_{A\in\underline{A}'}\hom(A, Y_*)$ is final, $g: Y_*\to X_*$ is monomorphism in $\underline{B}$. Hence $Y_*\in\underline{B}$. In all, $Y\in\mathrm{FIN}_S(\underline{A}:\underline{B})$.

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