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On some properties of G-ANR's

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Let G be a finite group throughout this paper. Bredon discussed CW-complexes on which G acts nicely, and called them G-complexes in [2]. After that many properties of G-complexes, G-ANR's and the relations between them corresponding the relations [5, Theorem 2.2.6] between CW-complexes and ANR's are discussed in [1, 4].

In this paper we examine some properties of G-ANR's corresponding those of ANR's based on the above results.

Definition A metrizable G-space X is called a G-ANR (a G-absolute neighborhood retract) if and only if X has the G-neighborhood extension property for all metrizable G-spaces, *i.e.*, any G-map $f: A \to X$ of every closed G-subspace A of every metrizable G-space Y can be extended equivariantly to an open G-neighborhood U of A in Y.

Theorem 1. *Let X be a G-ANR. Then any open G-subset A of X is a G-ANR.*

Proof. Let B be any closed G-subset of an arbitrary metrizable G-space Y, $f: B \to X$ be any G-map. Then the composition $i \circ f$: $B \to X$ of the given f and the inclusion $i: A \to X$ has a G-extension $F: V \to X$ over some G-neighborhood V of B in Y, since X is a G-ANR. Define $U = F^{-1}(A)$, then U is an open G-neighborhood of B in Y. Hence $F' = F|U: U \to A$ is a G-extension of f. So A is a G-ANR.

Homotopy extension property (or HEP) is discussed for many spaces. For example, if Y is an ANR, then the HEP says as following which is known as Borsuk's theorem: Let A be a closed subset of a metrizable space X. Then any map $f: (X \times \{0\}) \cup (A \times I) \to Y$ has an extension $f': X \times I \to Y$ [5, Theorem 2.1.3]. And for the space (X, A) of a pair of G-complex X and its G-subcomplex A, Bredon [2, Chap I, §1] shows the G-HEP for this (X, A) as following: Given a G-map $f: X \to Y$ and a G-homotopy $F: A \times I \to Y$ such that $F|A \times \{0\} = f|A$, then then there exists a G-homotopy $\tilde{F}$: $X \times I \to Y$ such that $\tilde{F}|X \times \{0\} = f$ and $\tilde{F}|A \times I = F$.

Now we show the G-HEP for the space Y which is a G-ANR and convex. The following definition is induced from [3].

Definition A G-subspace A of a G-space X is said to have the G-HEP in X with respect to a G-space Y if and only if every partial G-homotopy $h_t: A \to Y$, $(0 \leq t \leq 1)$, of an arbitrary G-map $f: X \to Y$ has a G-extension $f_t: X \to Y$, $(0 \leq t \leq 1)$, such that $f_0 = f$.

Theorem 2. *If Y is a G-ANR and convex, then every closed G-subspace A of an arbitrary* *metrizable G-space X has the G-HEP in X with respect to Y.*

Proof. Consider an arbitrary G-map $f:X\to Y$ defined on a metrizable G-space X and any partial G-homotopy $h_t:A\to Y$, $(0\leqq t\leqq 1)$, of f defined on a closed G-subspace A of X. Let $P=X\times I$, and $T=(X\times\{0\})\cup(A\times I)$, then T is closed in P. Define a G-map $H:T\to Y$ by

$$H(x,t)=\begin{cases} f(x) & (\text{if } x\in X \text{ and } t=0) \\ h_t(x) & (\text{if } x\in A \text{ and } t\in I) \end{cases}$$

Then, by Borsuk's theorem there exists an extension $H':P\to Y$ of H. Define a map $F:P\to Y$ by

$$F(x)=\frac{1}{|G|}\sum_{g\in G} gH'(g^{-1}x) \qquad \text{for } x\in P.$$

Then since $gH'(g^{-1}x)\in Y$, $\sum_{g\in G}\frac{1}{|G|}=1$, and Y is convex, $F(x)$ is a well defined G-map and a G-extension of H.

So, if we define $f_t:X\to Y$, $(0\leqq t\leqq 1)$, as $f_t(x)=F(x,t)$ for every $x\in X$ and every $t\in I$, then cleary $f_0=f$.

For every metrizable G-space Y, it can be imbedded as a closed G-subspace of a convex G-subset Z in the Banach G-space [4, Proposition 1.1]. Under this condition, we have the following.

Theorem 3. *Let Y be a metrizable G-space and imbedded as a closed G-subspace of Z which is metrizable. Then Y is a G-ANR if and only if for every G-deformation* $h_t: Y\to Y$, $(0\leqq t\leqq 1)$, *of Y, the G-map* h_1 *has a G-extension* $h: U\to Y$ *over some G-neighborhood U of Y in Z.*

Proof. Suppose that Y is a G-ANR. Then, since Y is a G-ANR and closed in the metrizable G-space Z, by the definition of a G-ANR any G-map $h_1: Y\to Y$ can be extended to $h: U\to Y$ over some G-neighborhood U of Y in Z.

Conversely, suppose that for any G-deformation h_t, h_1 has a G-extension. Then, take a G-deformation h_t such that h_1 be an identity on Y and let $h: V\to Y$ be a G-extension of h_1 over some G-neighborhood V of Y in Z. Let $f:A\to Y$ be any G-map of any closed G-subspace A of any metrizable G-space X. Since Z is convex in a Banach G-space, by [4, Proposition 1.2] it is a G-ANR. Hence the composition $\phi=i\circ f:A\to Z$ of the given G-map f and the inclusion $i: Y\to Z$ can be extended to $\psi: W\to Z$ over some G-neighborhood W of A in X. Let $U=\psi^{-1}(V)$. Then U is a G-neighborhood of A in X. Define a G-map $g: U\to Y$ by $g(x)=h\circ\psi(x)$ for every $x\in U$, then g is a G-extension of f. So, Y is a G-ANR.

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