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Note on the Gelfand Integral

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1. Introduction

A theory of integration similiar to the Bochner integral is impossible for functions that are only weak*-measurable.

Furthermore, it is impossible to use the Bochner integral theory directly to integrate a function f if $\|f\|$ is not integrable.

Nevertheless, there are simple thing available to integrate some such functions and as a small part of Gelfand contribution to functional analysis shows this simple method has some strong properties which will be presently investigated.

Let (Ω, Σ, μ) be a finite measure space and X a Banach space.

If $f:\Omega \to X^*$ is X-measurable, then f is called *weak*-measurable.*

Let f be a weak*-measarable function on Ω such that $xf \in L_1(\mu)$, for all $x \in X$, then the Gelfand integral of f over $E \in \Sigma$ is defined by the element x_E^* of X^* such that

$$x_E^*(x) = \int_E xf d\mu \text{ for all } x \in X.$$

2. Main theorems

Theorem 1. *Suppose f is weak*-measurable function on Ω and $xf \in L_1(\mu)$ for all x in X. Then for each $E \in \Sigma$ there exists the Gelfand integral of f over E.*

Proof. Let $E \in \Sigma$ and define $T: X \to L_1(\mu)$ by $T(x) = x(f\chi_E)$. Note that T is closed. Indeed, if $\lim_n x_n = x$ and $T(x_n) = g$ exists in $L_1(\mu)$, then some subsequence $x_{n_i}(f\chi_E) = T(x_{n_i})$ tends μ-almost everywhere to g.

But $\lim_n x_n(f\chi_E) = x(f\chi_E)$ everywhere. Hence $xf = g$ μ-almost everywhere and T is a closed linear operator.

It is easy to see from closed graph theorem that T is continuous. Hence $\|x(f)\| \leq \|T\| \cdot \|x\|$. Since the operation of integrating over E is continuous linear functional, it follows that $|\int_E xf d\mu| \leq \|T\| \cdot \|x\|$. Hence the mapping $x \to \int_E xf d\mu$ defines continuous linear functional on X.

Therefore there exist the element x_E^* of X^* such that $x_E^*(x) = \int_E xf d\mu$ for all $x \in X$, and $E \in \Sigma$.

Theorem 2. *If f is Gelfand integrable, then $\int_{(\cdot)} f d\mu$ is weak* countably additive vector measure on Σ.*

Proof. If (E_n) is a sequence of disjoint members of Σ, then

$$x\left(\int_{\bigcup_{n=1}^{\infty} E_n} f d\mu\right)=\int_{\bigcup_{n=1}^{\infty} E_n} x f d\mu=\sum_{n=1}^{\infty}\int_{E_n} x f d\mu$$

$$=\sum_{x=1}^{\infty} x\left(\int_{E_n} f d\mu\right).$$

References

〔1〕 D.L. Cohn. *Measure theory*. Bostion, Birkhäuser 1980.

〔2〕 D.B. Dimitron. A remark the Gelfand integral. *Functional Anal.* 5(1971), pp. 84-85.

〔3〕 J. Diestel and B. Faires. On vector measures. *Trans. Amer. Math. Soci,* 1908(1974) pp. 253-271.

〔4〕 G.A. Edgar. Measurability in a Banach space. *Indiana Univ. Math. J.* 26(1977) pp. 663-677.