Note on the Gelfand Integral

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1. Introduction

A theory of integration similar to the Bochner integral is impossible for functions that are only weak*-measurable.

Furthermore, it is impossible to use the Bochner integral theory directly to integrate a function $f$ if $\|f\|$ is not integrable.

Nevertheless, there are simple things available to integrate some such functions and as a small part of Gelfand contribution to functional analysis shows this simple method has some strong properties which will be presently investigated.

Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $X$ a Banach space.

If $f: \Omega \rightarrow X^*$ is $X$-measurable, then $f$ is called weak*-measurable.

Let $f$ be a weak*-measurable function on $\Omega$ such that $xf \in L_1(\mu)$, for all $x \in X$, then the Gelfand integral of $f$ over $E \in \Sigma$ is defined by the element $x_E: X^*$ such that

$$x_E^*(x) = \int_E x f d\mu$$

for all $x \in X$.

2. Main theorems

**Theorem 1.** Suppose $f$ is weak*-measurable function on $\Omega$ and $xf \in L_1(\mu)$ for all $x$ in $X$. Then for each $E \in \Sigma$ there exists the Gelfand integral of $f$ over $E$.

**Proof.** Let $E \in \Sigma$ and define $T: X \rightarrow L_1(\mu)$ by $T(x) = x(fx)$). Note that $T$ is closed. Indeed, if $\lim_n x_n = x$ and $T(x_n) = g$ exists in $L_1(\mu)$, then some subsequence $x_n(fx_n) = T(x_n)$ tends $\mu$-almost everywhere to $g$.

But $\lim_n x_n(fx_n) = x(fx)$ everywhere. Hence $xf=g \mu$-almost everywhere and $T$ is a closed linear operator.

It is easy to see from closed graph theorem that $T$ is continuous. Hence $\|x(f)\| \leq \|T\| \cdot \|x\|$.

Therefore there exist the element $x_E^*$ of $X^*$ such that $x_E^*(x) = \int_E x f d\mu$ for all $x \in X$, and $E \in \Sigma$.

**Theorem 2.** If $f$ is Gelfand integrable, then $\int_{\Sigma} f d\mu$ is weak* countably additive vector measure on $\Sigma$.

**Proof.** If $(E_n)$ is a sequence of disjoint members of $\Sigma$, then
\[ x \left( \bigcup_{n=1} x_{E_n} f d\mu \right) = \sum_{n=1} x \left( \bigcup_{E_n} x f d\mu \right) = \sum_{n=1} x_{E_n} x f d\mu \]

References