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On the Homology Modules of Differential Modules.

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0. Introduction

Let $_RM$ be the category of R-modules and R-homomorphisms, where R is a commutative ring with unity.

The differential modules in $_RM$ and some properties of them have been already introduced.

In this paper, we will discuss "Snake Lemma" which constructs the ker-coker sequence and using the Snake Lemma and some properties of the differential modules, we will construct the exact triangle of homology modules of differential modules.

Most of notations in this paper are taken from[4].

1. The construction of the ker-coker sequence (Snake Lemma)

Consider $h: X \to Y$ in $_RM$ and let A, B be submodules of X, Y respectively.

If $h(A) \subset B$, there exists a homomorphism $h^{\#}: X/A \to Y/B$ difined by $h^{\#}(u+A) = h(u) + B$ for all $u \in X$.

This homomorphism $h^{\#}$ is called the induced homomorphism of h on the quotient modules.

In particular, for $h: X \to Y$ in $_RM$, we have $\mathrm{coim}h = X/\ker h \cong \mathrm{im}\ h$. Snake Lemma may be proved by a series of the following propositions.

Proposition 1.1. *Given a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\xi"] \arrow[d, "h"'] & B' \arrow[d, "h'"] \\ A' \arrow[r, "\eta"] & B' \end{tikzcd}$$

in $_RM$, there exist homomorphisms $\ker h \xrightarrow{\xi^r} \ker h'$ and $\mathrm{coker}\, h \xrightarrow{\eta^{\#}} \mathrm{coker}\, h'$ defined by $\xi^r(x) = \xi(x)$ i.e. $\xi^r = \xi | \ker h$ and $\eta^{\#}(y + \mathrm{im}h) = \eta(y) + \mathrm{im}h'$ for all $x \in \ker h$ and $y \in A'$.

Proof. This follows from [2], p. 63 and [4], p. 39.

Proposition 1.2. *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \end{tikzcd}$$

be a commutative diagram with exact rows in $_RM$.

If f' is monic, then the induced sequence $\ker \alpha \xrightarrow{f^r} \ker \beta \xrightarrow{g^r} \ker \gamma$ is exact.

On the other hand if g is epic, then the resulting sequence $\operatorname{coker}\alpha \xrightarrow{f'^{\#}} \operatorname{coker}\beta \xrightarrow{g'^{\#}} \operatorname{coker}\gamma$ *is exact.*

Proof. This follows from [2], p. 63 and [4], p. 40.

Theorem 1.3. Snake Lemma *Consider the commutative diagram with exact rows:*

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & O \\ O \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

in $_RM$.

There exists an exact sequence

$\ker\alpha \xrightarrow{f^r} \ker\beta \xrightarrow{g^r} \ker\gamma \xrightarrow{h} \operatorname{coker}\alpha \xrightarrow{f'^{\#}} \operatorname{coker}\beta \xrightarrow{g'^{\#}} \operatorname{coker}\gamma$, *where* $h: c \longmapsto (f'^{-1}\circ\beta\circ g^{-1})(c) + im\alpha$.

Proof. This follows from [2], p. 63-p. 64, [4], p. 40-p. 41 and [3], p. 174.

This sequence is called the ker-coker sequence in [4] and [2].

2. The homology modules of differential modules.

By a differential module X, we mean a module X together with a given endomorphism $d: X \to X$ of the module X satisfying the condition $d\circ d=0$.

Then d is called a differentiation and the quotient module $H(X)=\ker d/\operatorname{im} d$ is defined because of $\operatorname{im} d \subset \ker d$.

We will call $H(X)$ the homology of the differential module X.

Consider an arbitrary differential module X with differentiation $d: X \to X$.

Then we define the following notations:

$$Z(X)=\ker d, \quad Z'(X)=\operatorname{coker} d = X/\operatorname{im} d$$
$$B(X)=\operatorname{im} d, \quad B'(X)=\operatorname{coim} d = X/\ker d.$$

([4], p. 53-p. 54)

Proposition 2.1. *Let X be a differential module with differentiation $d: X \to X$.*

Then (1) *the differentiation d induces an isormorphism $\delta: B'(X) \to B(X)$.*

(2) *d admits the following factorization:*

$$X \longrightarrow Z'(X) \longrightarrow B'(X) \xrightarrow{\delta} B(X) \longrightarrow Z(X) \longrightarrow X.$$

Furthermore, this factorization of d yields a homomorphism $d': Z'(X) \longrightarrow Z(X)$

(3) *$\ker d' = H(X) = \operatorname{coker} d'$.*

Proof (1). By the 1st isomorphism theorem, there exists an isomorphism

$$\delta: B'(X) = X/\ker d \longrightarrow B(X) = \operatorname{im} d.$$

(2) At the first, we take the natural projection

$$p: X \longrightarrow Z'(X) \text{ defined by } x \longmapsto x + \operatorname{im} d.$$

Then p is epic.

Let 1_X be the identity endomorphism of X. Since $1_X(\operatorname{im} d) = \operatorname{im} d \subset \ker d$, $1_X{}^{\#}: Z'(X) \to B'(X)$ is well defined by $1_X{}^{\#}(x+\operatorname{im} d) = x + \ker d$ for all $x \in X$.

Consider $i: B(X) \to Z(X)$ and $j: Z(X) \to X$ as inclusion homomorphisms. Thus d admits the following factorization:

$$X \xrightarrow{p} Z'(X) \xrightarrow{1_X{}^{\#}} B'(X) \xrightarrow{\delta} B(X) \xrightarrow{i} Z(X) \xrightarrow{j} X.$$

Also, we obtain a homomorphism $d': Z'(X) \longrightarrow Z(X)$, where $d' = i \circ \delta \circ 1_X{}^{\#}$ is defined by $d'(x+\mathrm{im}d) = d(x)$.

(3) Since j is an inclusion and p is epic, we have $\mathrm{im}d' = \mathrm{im}\ j \circ d' \circ p = \mathrm{im}d = B(X)$. Therefore coker $d' = Z(X)/\mathrm{im}d' = Z(X)/B(X) = H(X)$.

Also, $\ker d' = \{x+\mathrm{im}d \mid d'(x+\mathrm{im}d) = 0,\ x \in X\} = \{x+\mathrm{im}d \mid d(x) = 0,\ x \in X\} = \{x+\mathrm{im}d \mid x \in \ker d\} = Z(X)/B(X) = H(X)$.

Theorem 2.2. *For every short exact sequence*

$$O \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow O$$

of homomorphisms of differential modules which commute with the differentiations d_1, d_2 and d_3, that is, for the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "d_1"'] & B \arrow[r, "g"] \arrow[d, "d_2"'] & C \arrow[r] \arrow[d, "d_3"'] & O \\
O \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C &
\end{tikzcd}$$

(1) *we obtain a commutative diagram*

$$\begin{tikzcd}
 & Z'(A) \arrow[r, "f^{\#}"] \arrow[d, "d_1'"'] & Z'(B) \arrow[r, "g^{\#}"] \arrow[d, "d_2'"'] & Z'(C) \arrow[r] \arrow[d, "d_3'"'] & O \\
O \arrow[r] & Z(A) \arrow[r, "f^r"] & Z(B) \arrow[r, "g^r"] & Z(C) &
\end{tikzcd}$$

of homomorphisms of modules with exact rows, and

(2) *we obtain an exact sequence*

$$H(A) \longrightarrow H(B) \longrightarrow H(C) \xrightarrow{h^{\#}} H(A) \longrightarrow H(B) \longrightarrow H(C).$$

This sequence is called the exact triangle of homology modules of differential modules:

$$\begin{tikzcd}
H(A) \arrow[rr] & & H(B) \arrow[dl] \\
 & H(C) \arrow[ul] &
\end{tikzcd}$$

Proof. (1) Since g is epic, $Z'(A) \xrightarrow{f^{\#}} Z'(B) \xrightarrow{g^{\#}} Z'(C)$ is exact, where $f^{\#}(a+\mathrm{im}d_1) = f(a) + \mathrm{im}d_2$ and $g^{\#}(b+\mathrm{im}d_2) = g(b) + \mathrm{im}d_3$ for all $a \in A$ and $b \in B$. Since f is monic, $Z(A) \xrightarrow{f^r} Z(B) \xrightarrow{g^r} Z(C)$ is exact, where $f^r = f|_{Z(A)}$ and $g^r = g|_{Z(B)}$.

By proposition 2.1 (2), we can define d_1', d_2' and d_3' as the followings:

$d_1'(a+\mathrm{im}d_1) = d_1(a)$, $d_2'(b+\mathrm{im}d_2) = d_2(b)$,

$d_3'(c+\mathrm{im}d_3) = d_3(c)$ for all $a \in A$, $b \in B$ and $c \in C$.

It is easy to verify that the diagram is commute:

$$\begin{tikzcd}
Z'(A) \arrow[r, "f^{\#}"] \arrow[d, "d_1'"'] & Z'(B) \arrow[r, "g^{\#}"] \arrow[d, "d_2'"'] & Z'(C) \arrow[d, "d_3'"'] \\
Z(A) \arrow[r, "f^r"] & Z(B) \arrow[r, "g^r"] & Z(C).
\end{tikzcd}$$

We will show that $g^{\#}$ is epic and f^{r} is monic.

Let $c+\mathrm{im}d_3 \in Z'(C)$ with $c \in C$. Since g is epic, there exists $b \in B$ with $g(b)=c$. Hence $g^{\#}(b+\mathrm{im}d_2)=g(b)+\mathrm{im}d_3=c+\mathrm{im}d_3$ and thus $g^{\#}$ is epic.

Let $a \in \ker f^{r}$. Then $f^{r}(a)=f(a)=0$. Since f is monic, $a=0$ and so f^{r} is monic.

(2) By theorem 1.3, there is an exact sequence

$$\ker d_1' \to \ker d_2' \to \ker d_3' \to \mathrm{coker} d_1' \to \mathrm{coker} d_2' \to \mathrm{coker} d_3'$$

By proposition 2.1(3), we obtain an exact sequence

$$H(A) \longrightarrow H(B) \longrightarrow H(C) \longrightarrow H(A) \longrightarrow H(B) \longrightarrow H(C).$$

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