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A Study on *z*-*S*-closed Spaces

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Abstract: In this paper, we define the z-S-closed spaces using the notions of zero-sets and S-closed spaces introduced by T. Thompson, and investigate some properties of these spaces. We also obtain the following results.

If a space X is z-S-closed, then every cover of z-regular semiopen sets has a finite proximate subcover. A z-extremally disconnected z-QHC space is z-S-closed. z-S-closed is contagious.

1. Preliminary definitions.

A subset Z of X is called a zero-set if $Z=\{x\in X \mid f(x)=0\}$ for some continuous function $f:X\to R$, where X, R will represent, respectively, the topological space, the set of real numbers. Now, we consider the definitions of several basic terminologies using the concept of zero-set.

Definition 1.1. A subset E of X is said to be *z-open* if for every point x of E, there exists a zero-set nbd(abbreviation of neighborhood) N of x such that $x\in N\subset E$.

Definition 1.2. If E is a subset of X, then *z-closure* of E in X is the set $z\text{-}cl\ E=\{x\in X \mid$ *for every zero-set nbd* N *of* x, $N\cap E\neq\phi\}$.

Definition 1.3. A subset E of X is said to be *z-closed* if $z\text{-}cl\ E=E$.

Definition 1.4. Let E be a subset of X. A point x is said to be *z-interior point* of E if there exists a zero-set nbd N of x such that $x\in N\cap E$. The set of z-interior point of E is called the *z-interior* of E, written $z\text{-}int\ E$.

Definition 1.5. A family $\{V_\alpha \mid \alpha\in\mathscr{A}\}$ of z-open subsets of X is said to be *z-open cover* of X if $\bigcup_{\alpha\in\mathscr{A}} V_\alpha=X$.

2. *z*-*S*-closed spaces and other spaces.

Definition 2.1. A space X is *z-quasi-H-closed* (denoted z-QHC) if every z-open cover has a finite proximate subcover (every z-open cover has a finite subfamily whose closures cover the space).

Definition 2.2. A set E in a space X is *z-semiopen* if $z\text{-}int\ E\subset E\subset z\text{-}cl\ (z\text{-}int\ E)$.

Definition 2.3. A space X is *z-S-closed* if every z-semiopen cover has a finite proximate subcover.

It is obvious that every z-S-closed space is z-QHC but the converse is not true.

Definition 2.4. A subset E of X is said to be *z-regular open* if $E=z\text{-}int\ (z\text{-}cl\ E)$.

Definition 2.5. A subset F of X is said to be *z-regular closed* if $F=z\text{-}cl\ (z\text{-}int\ F)$.

Definition 2.6. A subset E of X is *z-regular semiopen* if there exists a z-regular open set U such that $U\subset E\subset z\text{-}cl\ U$.

Therorem 2.7. *If a space X is z-S-closed, then every cover of z-regular semiopen sets has a finite* *proximate subcover.*

Proof: The result follows from the fact that every z-regular semiopen set is z-semiopen.

Definition 2.8. A set E in a space X is *strongly z-semiopen* if E is z-semiopen and $z\text{-}cl\ E = z\text{-}cl\ (z\text{-}int\ (z\text{-}cl\ E))$.

Lemma 2.9. *If a subset E of X is strongly z-semiopen, then z-int (z-cl E) is z-regular semiopen.*

Proof: The result is immediate from definition 2.8.

Definition 2.10. A space X is *strongly z-S-closed* if every strongly z-semiopen cover has a finite proximate subcover.

Theorem 2.11. *If every cover of z-regular semiopen sets has a finite proximate subcover, then a space X is strongly z-S-closed.*

Proof: If the space is not strongly z-S-closed then there is a strongly z-semiopen cover $\{V_\alpha | \alpha \in \mathscr{A}\}$ which has no finite proximate subcover. Then since $z\text{-}int\ (z\text{-}cl\ V_\alpha) \subset z\text{-}int(z\text{-}cl\ V_\alpha) \cup V_\alpha \subset z\text{-}cl\ (z\text{-}int(z\text{-}cl\ V_\alpha))$, $\{z\text{-}int(z\text{-}cl\ V_\alpha) \cup V_\alpha | \alpha \in \mathscr{A}\}$ is a z-regular semiopen cover which has no finite proximate subcover.

Definition 2.12. A space X is *z-extremally disconnected* if z-regular closed subsets of X is z-open.

Theorem 2.13. *A z-extremally disconnected z-QHC space is z-S-closed.*

Proof: If X is z-extremally disconnected and $E \subset X$, then since $z\text{-}int(z\text{-}cl\ E) = E$, E is z-open, and since $(z\text{-}int(z\text{-}cl\ E))^c = E^c$, i.e, $z\text{-}cl(z\text{-}int\ E^c) = E^c$, E^c is z-open and so E is z-closed. Hence the z-regular open sets are z-clopen. On one had, since E is z-regular semiopen, there exists z-regular open set U such that $U \subset E \subset z\text{-}cl\ U$. Hence $U = E = z\text{-}cl\ U$. Thus z-regular semiopen sets are z-clopen, where E^c is the complement of E.

Lemma 2.14. *If a subset E of X is z-regular closed, then E is z-regular semiopen.*

Proof: Since E is z-regular closed, $z\text{-}cl(z\text{-}int\ E) = E$, i.e, $z\text{-}int(z\text{-}cl(z\text{-}int\ E)) = z\text{-}int\ E$. This implies that $z\text{-}int\ E$ is z-regular open. Hence $z\text{-}int\ E \subset E \subset z\text{-}cl(z\text{-}int\ E)$. Thus E is z-regular semiopen.

Theorem 2.15. *If a subset E of a z-S-closed X is z-clopen, then E is z-S-closed.*

Proof: Let $\{V_\alpha | \alpha \in \mathscr{A}\}$ be a z-semiopen cover of E and let $E^c = V_{\alpha_0}$. Then $X = (\bigcup_{\alpha \in \mathscr{A}} V_\alpha) \cup V_{\alpha_0}$. Since X is z-S-closed, there exists a finite subfamily $\{V_{\alpha_i} | i = 1, 2, \cdots n\}$ of $\{V_\alpha | \alpha \in \mathscr{A}\}$ such that $X \subset (\bigcup_{i=1}^{n} z\text{-}cl\ V_{\alpha_i}) \cup z\text{-}cl\ V_{\alpha_0}$. On the other hand, since V_{α_0} is z-clopen, $z\text{-}cl\ V_{\alpha_0} = V_{\alpha_0}$ and so $E \subset \bigcup_{i=1}^{n} z\text{-}cl\ V_{\alpha_i}$. Hence E is z-S-closed.

Corollary 2.16. *If a subset E of a z-S-closed X is z-regular closed, then z-int E is strongly z-S-closed.*

Proof: Let $\{V_\alpha | \alpha \in \mathscr{A}\}$ be a z-regular semiopen cover of E and let $E^c = V_{\alpha_0}$. Then $X = (\bigcup_{\alpha \in \mathscr{A}} V_\alpha) \cup (V_{\alpha_0})$. Since X is z-S-closed, there exists a finite subfamily $\{V_{\alpha_i} | i = 2, \cdots n\}$ of $\{V_\alpha | \alpha \in \mathscr{A}\}$ such that $X \subset (\bigcup_{i=1}^{n} z\text{-}cl\ V_{\alpha_i}) \cup z\text{-}cl\ V_{\alpha_0}$ and hence $z\text{-}int\ E \subset \bigcup_{i=1}^{n} z\text{-}cl\ V_{\alpha_i}$. Thus $z\text{-}int\ E$ is strongly z-S-closed.

Corollary 2.17. *If subsets E, F of a z-S-closed X is z-S-closed, then $E\cup F$ is z-S-closed.*

Proof: Let $\{V_\alpha|\alpha\in\mathcal{A}\}$ be a z-semiopen cover of $E\cup F$. Then $E\cup F\subset\bigcup_{\alpha\in\mathcal{A}} V_\alpha$. Since E is z-S-closed, there exists a finite subfamily $\{V_{\alpha_i}|i=1,2,\cdots n\}$ of $\{V_\alpha|\alpha\in\mathcal{A}\}$ such that $E\subset\bigcup_{i=1}^{n} z\text{-}cl\ V_{\alpha_i}$ and since F is z-S-closed, there exists a finite subfamily $\{V_{\alpha_k}|k=m+1,\cdots,m+n\}$ of $\{V_\alpha|\alpha\in\mathcal{A}\}$ such that $F\subset\bigcup_{k=m+1}^{n} z\text{-}cl\ V_{\alpha_k}$. Hence $E\cup F\subset(\bigcup_{i=1}^{m} z\text{-}cl\ V_{\alpha_i})\cup(\bigcup_{k=m+1}^{n} z\text{-}cl\ V_{\alpha_k})$, i.e, $E\cup F\subset\bigcup_{l=1}^{m+n} z\text{-}cl\ V_{\alpha_l}$. Thus $E\cup F$ is z-S-closed.

Lemma 2.18. *Let E be a subset of X. Then $cl(z\text{-}cl\ E)=z\text{-}cl\ E$.*

Proof: Suppose that there exists $x\in X$ such that $x\in cl(z\text{-}cl\ E)$ but $x\notin z\text{-}cl\ E$. Since $x\notin z\text{-}cl\ E$, there exists a zero-set nbd N of x such that $N\cap E=\phi$. On the while, $int N\cap z\text{-}cl\ E\neq\phi$. Take $x'\in int\ N\cap z\text{-}cl\ E$. Then for every zero-set nbd M of x', $M\cap E\neq\phi$ since $x'\in z\text{-}cl\ E$. On the other hand, $N\cap M$ is a zero-set nbd of x', and since $N\cap E=\phi$, $(N\cap M)\cap E=\phi$. This contradiction proves the lemma.

Theorem 2.19. *z-S-closed is contagious.*

(A property R is contagious if a space has the property whenever a dense subset has the property.)

Proof: Let E be a z-S-closed dense subset of X. If $\{V_\alpha|\alpha\in\mathcal{A}\}$ is a z-semiopen cover of X, then $\{V_\alpha\cap E|\alpha\in\mathcal{A}\}$ is a z-semiopen cover of E. Hence there exists a finite subfamily $\{V_{\alpha_k}\cap E|k=1,2,\cdots n\}$ of $\{V_\alpha\cap E|\alpha\in\mathcal{A}\}$ such that $E\subset\bigcup_{k=1}^{n} z\text{-}cl(V_{\alpha_k}\cup E)\subset\bigcup_{k=1}^{n} z\text{-}cl\ V_{\alpha_k}$. On one hand, $X=clE\subset cl\bigcup_{k=1}^{n} z\text{-}cl V_{\alpha_k}=\bigcup_{k=1}^{n} z\text{-}cl\ V_{\alpha_k}$. Thus X is z-S-closed.

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