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—Research Paper—

Quadratic Complementary Programming

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Abstract

The present paper provides a method for solving a complementary programming problem with quadratic objective function subject to linear constraints. The procedure developed is based on the simplex method for quadratic programming problem. An example is added to illustrate the procedure.

1. Introduction

This paper is concerned with solution of a mathematical programming problem called complementary programming problem with indefinite quadratic objective function. The problem is stated as follows:

$$\text{Minimize } Z=(d'x+e'u+f'v+\alpha)(r'x+s'u+t'v+\beta) \tag{1.1}$$

Subject to

$$Ax+Bu+Cv=b \tag{1.2}$$

$$u \cdot v=0 \tag{1.3}$$

$$x, u, v \geqslant 0 \tag{1.4}$$

where x, u, v are n, m and m dimensional vectors of variables respectively; $d, r; e, s; f, t$ are n, m and m dimensional vectors of constants respectively; A, B and C are $p \times n$, $p \times m$ and $p \times m$ matrices of constants respectively; b is a P^- dimensional vectors of constants; α, β are scalars and $\{'\}$ denotes the transpose of a vector.

Further, it is assumed that

(a) $(d'x+e'u+f'v+\alpha)$ and $(r'x+s'u+t'v+\beta)$ are positive for all feasible solutions.

(b) The set of feasible solutions is bounded, closed convex polyhedron.

(c) Solution set contains at least two different points.

In the absence of the condition $u \cdot v=0$, the present problem is a quadratic programming problem with indefinite quadratic objective function, and such type of problems can be dealt with by a simplex method by Candler and Townsley [2] and Swarup [6].

The condition $u \cdot v=0$ permits solutions in which either $u_j=0$ or $v_j=0$ or both all $j=1, 2, \cdots,$

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m. The ordinary complementary programming problem may be stated as follows.

$$\text{Minimize } Z = dx + eu + fv + \alpha$$
$$\text{Subject to } Ax + Bu + Cv \leq b$$
$$u \cdot v = 0$$
$$x, u, v \geqslant 0$$

A branch and bound technique for solving this type of problem is given by Ibaraki [4]. This type of problems are met in various fields of Operations Research such as absolute value programming, 0–1 mixed integer programming.

Our main aim is to develop a method exactly similar to 'simplex technique' for solving the problem considered in this paper, which should be well adapted to high speed computation.

2. Some Definitions

(a) **Feasible Solution:** Any set of variables (x, u, v) satisfying the conditions (1.2), (1.3) and (1.4) is a feasible solution.

(b) **Optimal Solution:** A feasible solution that minimize the objective function (1.1) is an optimal solution.

(c) **Partial Solution:** A partial solution S is an assignment of zeros to a subset of variables u and v in which u_j and v_j never appears simultaneously. We adopt the notational convention that $j \in S$ implies $u_j = 0$ and $-j \in S$ implies $v_j = 0$. For example, $S = (4, -3, -2)$ means $u_4 = 0$, $v_3 = 0$ and $v_2 = 0$.

The variables assigned in partial solution S are called nullified variables of S.

Any variables u_j and v_j such that neither j nor $-j$ is included in S are free.

A partial solution T is known as continuation of partial solution S if $S \subset T$ but if T includes j and $-j$ for all j then T is said to be a completion of S.

Thus for problem P and a partial solution S, partial problem $P(S)$ may be defined as follows:

Problem $P(S)$:

$$\text{Minimize } Z = (d'x + e'u + f'v + \alpha)\ \ (r'x + s'u + t'v + \beta)$$
$$\text{Subject to } Ax + Bu + Cv \leq b$$
$$u \cdot v = 0$$
$$x, u, v \geqslant 0$$
$$u_j = 0, \text{ if } j \in S; \ v_j = 0 \text{ if } -j \in S$$

3. Theoretical Development

For each partial solution $P(S)$, consider another problem $P'(S)$ without the complementary condition $u \cdot v = 0$ as follows:

Problem $P'(S)$:

Minimze $Z=(d'x+e'u+f'v+\alpha) \quad (rx+su+tv+\beta)$

Subject to $Ax+Bu+Cv\leq b$

$x, u, v \geqslant 0$

$u_j=0$ if $j\in S$; $v_j=0$ if $-j\in S$

The problem $P'(S)$ is a quadratic programming problem with indefinite quadratic objective functions. This problem can be solved by the simplex technique given by Swarup [6].

We know that for a given basic feasible solution $WH=H^{-1}b$ with

$$Z_0=(D'HWH+\alpha) \quad (R'HWH+\beta)$$

to the problem $P'(S)$, where H is the basis matrix of p columns formed from the columns of matrix F (a matrix of constants of order $p\times(n+2m+p)$)

Let $Z_1=D'HWH+\alpha$

and $Z_2=R'HWH+\beta$

where $D'H$ and $R'H$ are the vectors having their components as the coefficients associated with the basic variables in objective function of $P'(S)$, : Further assume that for this basic feasible solution

$$t_j=H^{-1}a_j$$
$$Z_j^{(1)}=D'Ht_j$$
$$Z_j^{(2)}=R'Ht_j$$

are known for every column of a_j of F not in H.

As in (Swarup, 6), we have

$$\varDelta_j=Z_2(d_j-Z_j^{(1)})+Z_1(r_j-Z_j^{(2)})+\theta_j(d_j-Z_j^{(1)}) \quad (r_j-Z_j^{(2)})$$

where $Z_j^{(1)}$ and $Z_j^{(2)}$ are refer to the original basic feasible solution and θ_j is a scalar quantity. Thus for a given basic feasible solution $WH=H^{-1}b$, if for any column a_j in F but not in H, $\varDelta_j<0$ holds, and if at least one $t_j>0$, then it is possible to obtain a new basic feasible solution by replacing one of columns in H by a_j and the new value of the objective function satisfies:

$$\widehat{Z}_1\cdot\widehat{Z}_2\leq Z_1\cdot Z_2$$

If degeneracy is not present, $\widehat{Z}_1\cdot\widehat{Z}_2<Z_1\cdot Z_2$. Thus, we can move from one basic feasible solution to another by changing one vector at a time so long as there is some a_j not in the basis with the condition $\varDelta_j<0$, and at each step objective function is improved.

The process may not continue indefinitely because there are only finite number of basis and in the absence of degeneracy no basis can ever be repeated, and at the same time minimum is to occur at one of the basic feasible solutions. The process will terminate at the stage when all $\varDelta_j\geq 0$ for all columns of F.

The optimal solution of $P'(S)$ and the simplex tableau generated for its computation will provide us information about the optimal solution of $P(S)$. The following points may be taken into consideration:

(a) If $P'(S)$ has no feasible solution, $P(S)$ also has no feasible solution.

(b) If an optimal solution of $P'(S)$ satisfies the complementary condition $u\cdot v=0$, then it is clearly the optimal solution for $P(S)$.

(c) Let the value of the objective function at the optimal solution of $P'(S)$ is $Z_{p'}(S)$ and of $P(S)$ is $Z_p\ (S)$, if it exist. Then $Z_{p'}(S)\leq Z_p\ (S)$.

4. Algorithm

The problem $P'(S)$ is solved by the method given by Swarup [6]. If at the optimal solution the condition $u \cdot v = 0$ is satisfied, the process will terminate and the current optimal solution will be the optimal solution of the problem $P(S)$. But if the condition $u \cdot v = 0$ is not satisfied, then to determine the new partial problem calculate the penalty functions $p(u_j)$ and $p(v_j)$ of u_j and v_j (see appendix). Let $p(v_j) < p(u_j)$, we choose v_j to be departing variable from the basis, i.e. the next partial problem to be examined. Driving v_j out of the basis causes a transformation of the table and after that, the column corresponding to this is deleted. The increase in the objective value caused by the condition $v_j = 0$ can be easily calculated. The algorithm can be terminated whenever $Z_p(S) \geqslant Z_{p'}(S)$, where $Z_{p'}(S)$ is the incumbent value and $Z_p(S)$ is the objective value of $P(S)$.

5. Numerical Example:

Min. $Z = (2x_1 + x_2 + 2)\ (5 - 3x_1 - x_2)$

Subject to $2x_1 + 2x_2 \leq 7$

$x_1 - 4x_2 \leq 1$

$-2x_1 + 2x_2 \leq 1$

$x_1, x_2 \geq 0$

or Min. $Z = \{2 - (-)x_1 - (-1)x_2\}\ \{5 - (3)x_1 - (1)\ x_2\}$

Subject to $2x_1 + 2x_2 + x_3 = 7$

$x_1 - 4x_2 + x_4 = 1$

$-2x_1 + 2x_2 + x_5 = 1$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

The initial simplex table is shown below:

Table 1

Δ_j $\alpha_{0j}\ \beta_{00} - \beta_{0j}\ \alpha_{00}$	-16 x_1	-7 x_2	
$\alpha_{00}=2$	-2	-1	
$\beta_{00}=5$	3	1	$Z=10$
x_1	-1	0	
x_2	0	-1	
$x_3=7$	2	2	
$x_4=1$	$\boxed{1}$	4	
$x_5=1$	-2	7	

Table 2

Δ_j	16 x_4	-70 x_2	
$\alpha_{00}=4$	2	9	
$\beta_{00}=2$	-3	13	$Z=8$
$x_1=1$	1	-4	
x_2	0	-1	
$x_3=5$	-2	$\boxed{10}$	
x_4	-1	0	
$x_5=3$	2	6	

This is the optimal solution to the indefinite quadratic program without the complementary condition. Now, we shall impose the condition $x_1 \cdot x_2 = 0$. For this, we shall calculate the penalty of x_1 and x_2 as follows:

Table 3

Δ_j	5/2	7	
	x_4	x_3	
$\alpha_{00}=17/2$	1/5	9/10	
$\beta_{00}=-9/2$	−2/5	−13/10	$Z=-153/4$
$x_1=3$	1/5	4/10	
$x_2=1/2$	−1/5	1/10	
x_3	0	−1	
x_4	−1	0	
$x_5=6$	4/5	6/10	

Table 4

Δ_j	x_4	x_2	
$\alpha_{00}=4$	2	9	
$\beta_{00}=2$	−3	13	$Z^*=8$
$x_1=1$	1	−4	
x_2	0	−1	
$x_3=5$	−2	10	
x_4	−1	0	
$x_5=3$	2	−6	

$$a'_{14}=\frac{WH_1\cdot\beta_{01}}{\beta_{00}}+a_{11}=\frac{3(-2/5)}{-9/2}+1/5=7/15$$

$$a'_{13}=\frac{WH_1\cdot\beta_{02}}{\beta_{00}}+a_{12}=\frac{3(-13/10)}{-9/2}+4/10=19/15$$

Here $a'_{13}>a'_{14}$ Thus a'_{13} will be considered for calculation panalty function of x_1.

$$a'_{24}=\frac{WH_2\cdot\beta_{01}}{\beta_{00}}+a_{21}=\frac{1/2(-2/5)}{-9/2}-1/5$$

$$=-7/45$$

$$a'_{23}=\frac{WH_2\cdot\beta_{02}}{\beta_{00}}+a_{22}=\frac{1/2(-13/10)}{-9/2}+1/10$$

$$=11/45$$

$$a'_{23}>a'_{24}$$

$$p(x_1)=\frac{WH_1}{\beta_{00}}\times\frac{1}{\beta_{00}}\times\frac{\Delta_j}{a'_{13}}$$

$$=\frac{3}{-9/2}\times(-2/9)\times\frac{7}{19/15}$$

$$=\frac{4\times35}{9\times19}=\frac{140}{171}$$

$$p(x_2)=\frac{WH_2}{\beta_{00}}\times\frac{1}{\beta_{00}}\times\frac{\Delta_j}{a'_{23}}$$

$$=\frac{1/2}{-9/2}\times(-2/9)\times\frac{7}{11/45}$$

$$=\frac{2\times35}{9\times11}=\frac{70}{99}$$

Now $p(x_2)<p(x_1)$

Since $Z_2=-\frac{17}{2}\times\frac{9}{2}=-153/4$ and $p(x_1)=0.80$

Then $\frac{-153}{4}-0.80=-$ve which is less than the value of Z^*.

Hence $Z^*=8$ is the optimal solution

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Appendix

If the optimal solution of $P'(s)$ does not satisfy the conditions $u \cdot v = 0$, then to determine the next partial problem, we calculate penalty functions $p(u_j)$ and $p(v_j)$ as follows:

Let r the row in the optimal simplex table of $P'(s)$ corresponds to the basic variable u_j. Then $p(u_j)$ is determined as follows:

$$p(u_j) = \frac{\mathrm{WH}_r}{(Z_2)^2} \times \frac{\Delta' s}{a'_{rs}}$$

Where $\Delta' s = -\Delta s$

$$\Delta' s / a'_{rs} = \mathrm{Max}\ \{\Delta'_j / a_{rj} : a'_{rj} > 0\}$$

and $a'_{rs} = \dfrac{\mathrm{WH}_r (r_j - Z^{(2)})}{Z^{(2)}} + a_{rs}$

Here $a_s = (a_{1s}, a_{2s}, \cdots, a_{ps})$

and $a_s = H^{-1} \mu_s;\ s = 1, 2, \cdots, n + 2m + p$

$a'_{rs} = S_s$ [Hadley 3, Swarup 7]

Similarly penalty function for v_j can be calculated. As a result $Z_{p'}(s) - p(u_j)$ can be used as a lower bound of the objective value for $P(s, j)$ and $Z_{p'}(s) - p(v_j)$ as a lower bound of the objective value for $P(s, -j)$.

If $p(u_j) < p(v_j)$, then we choose u_j to be the departing variable from the basis. To derive the variable out of the basis, a pivot operation is performed and the optimal solution is obtained. After that, the column corresponding to the non-basic variable u_j is deleted. The solution is stored as the incumbent whose objective value is $Z_p(s)$ and value of $Z_{p'}(s) - p(u_j)$ is also calculated. If $Z_{p'}(s) - p(u_j) < Z_p(s)$, then $S = \{1\}$ does not lead to any better feasible solution than the incumbent and the process terminates. If these exist a row in the simplex table for $P'(s)$ such that $a'_{rs} \leq 0$ for $s = 1, 2, \cdots\cdots\cdots, x + 2m + p$ and $\mathrm{WH}_r > 0$, then u_j can not be nullified and we proceed to nullify v_j. This is because u_j must assume a positive value under the above condition. Similarly for v_j.