⟨Research Paper⟩

Journal of the Korean
Statistical Society
Vol. 11, No. 2, 1982

A Local Limit Theorem for Large Deviations

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ABSTRACT

A local limit theorem for large deviations for the i.i.d. random variables was given by Richter(1957), who used the saddle point method of complex variables to prove it. In this paper we give an alternative form of local limit theorem for large deviations for the i.i.d. random variables which is essentially equivalent to that of Richter. We prove the theorem by more direct and heuristic method under a rather simple condition on the moment generating function (m.g.f.). The theorem is proved without assuming that $E(X_i)=0$.

1. Introduction

Let X_1, X_2, $\cdots$, be a sequence of independent and identically distributed random variables with common distribution function F. Since the original work of Cramèr(1938), there have been a great deal of investigations on limiting probability for large deviations, namely, $Pr(X_1+\ldots+X_n\geq\sqrt{n}x_n)$ where $\{x_n\}$ is a sequence of constants increasing without bound. See Bahardur and Zabell(1979) and Nagaev(1979) and references therein.

In contrast with integral limit theorems, local limit theorems have received less attention. In fact the pioneering work by Richter(1957) still remains more or less intact except for a few minor modifications and extensions.

In this paper, we also consider Richter's paper and obtain an alternative but essentially equivalent form of his result. Our proof is more direct and shorter, and explains

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**This work was supported by the Ministry of Education, through the Research Institute of Basic Sciences, Seoul National University.

the way how the large deviation rate (or Choernoff index) should appear in local limit theorem as it does in integral limit theorem. Therefore we consider our proof more heuristic. The theorem is proved without assuming that $E(X_i)=0$.

2. Main Result

Theorem. Let X_1, X_2, $\cdots$ be i.i.d. with common distribution function F. Let m be a real number. Let the following conditions hold:

1) $\phi(s)=\int_{-\infty}^{\infty}\exp(sx)dF(x)<\infty$ for $s\in N$,
where N is an open interval containing zero.

2) There is a $\tau\in N$ such that

$$\phi'(\tau)/\phi(\tau)=m.$$

3) There is a positive integer n_0 such that

$$\sup_{s\in N}\int_{-\infty}^{\infty}\left|\frac{\phi(s+it)}{\phi(s)}\right|^{n_0}dt<\infty.$$

Let $f_n(\cdot)$ be the p.d.f. of $\frac{X_1+\cdots+X_n}{n}$ for $n\geq n_0$.

Then, for $\omega=\omega_n=o(1)$, we have, as $n\to\infty$,

$$f_n(m+\omega)=\frac{\sqrt{n}}{\sqrt{2\pi}\cdot\sigma}\exp\{-n\gamma(m+\omega)\}\{1+o(1)\},$$

where

$$\gamma(a)=\sup_{s\in N}\{sa-\log\phi(s)\}$$

and

$$\sigma^2=\frac{\phi''(\tau)\phi(\tau)-\phi'(\tau)^2}{\phi(\tau)^2}>0.$$

3. Proof of Theorem

We need the following lemmas to prove the theorem.

Lemma 1: Let $F_\tau(\cdot)$ be a d.f. with finite m.g.f. $\phi(\cdot)$ on some neighborhood N of zero. For each $\tau\in N$, let $F_\tau(\cdot)$ be defined as

$$dF_\tau(x)=\frac{\exp(\tau x)}{\phi(\tau)}dF(x).$$

Let $F_\tau^{(*n)}(\cdot)$ and $F^{(*n)}(\cdot)$ be n-fold convolution of $F_\tau(\cdot)$ and $F(\cdot)$ respectively.

Then

(1) $$dF_\tau^{(*n)}(x)=\frac{\exp(\tau x)}{\phi^n(\tau)}dF^{(*n)}(x).$$

Proof:

$$\int_{-\infty}^{\infty}\exp\,(itx)dF_{\tau}^{(*n)}(x)$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\{it(x_1+\cdots+x_n)\}dF_{\tau}(x_1)\cdots dF_{\tau}(x_n)$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\{it(x_1+\cdots+x_n)+\tau(x_1+\cdots+x_n)\}\phi^{-n}(\tau)dF(x_1)\cdots dF(x_n)$$
$$=\int_{-\infty}^{\infty}\exp(itx)\exp(\tau x)\phi^{-n}(\tau)dF^{(*n)}(x).$$

By the uniqueness of characteristic function, we have

$$dF_{\tau}^{(*n)}(x)=\frac{\exp(\tau x)}{\phi^{n}(\tau)}dF^{(*n)}(x)$$

as required.

We need another lemma which uses analytic property of m.g.f. $\phi(z)$.

Lemma 2: Let $\phi(z)$ be analytic, and let

$$\int_{-\infty}^{\infty}|\phi(s+it)|^{p}dt \quad (p>1)$$

exist and bounded for $s_1\leq s\leq s_2$.

Then as $t\to\pm\infty$,

$$\phi(s+it)\to 0 \text{ uniformly for } s_1+\delta\leq s\leq s_2-\delta(\delta>0).$$

Proof: Let $s_1+\delta\leq x\leq s_2-\delta$.

Then for $0<\rho\leq\delta$ and $z=x+iy$,

$$\phi(z)=\frac{1}{2\pi i}\int_{|\omega-z|=\rho}\frac{\phi(\omega)}{\omega-z}d\omega$$
$$=\frac{1}{2\pi}\int_{0}^{2\pi}\phi(z+\rho e^{i\theta})d\theta.$$

Hence $\frac{1}{2}\delta^2\phi(z)=\frac{1}{2\pi}\int_0^{\delta}\int_0^{2\pi}\phi(z+\rho e^{i\theta})\rho d\rho d\theta$ and

$$\frac{1}{2}\delta^2|\phi(z)|\leq\frac{1}{2\pi}\left\{\int_0^{\delta}\int_0^{2\pi}|\phi|^{p}\rho d\rho d\theta\right\}^{1/p}\left\{\int_0^{\delta}\int_0^{2\pi}\rho d\rho d\theta\right\}^{1-\frac{1}{p}}$$
$$\leq K(\delta)\left\{\int_{s_1}^{s_2}\int_{y-\delta}^{y+\delta}|\phi(s+it)|^{p}dtds\right\}^{1/p}.$$

Now $\int_{y-\delta}^{y+\delta}|\phi(s+it)|^{p}\,dt$ is bounded for $s_1\leq s\leq s_2$, and tends to 0 as $y\to\infty$, for every s. By the bounded convergence, the right hand side goes to zero as $y\to\infty$. Hence the result follows.

We are now ready to prove the theorem.

Recall that d.f. $F_s(\cdot)$ is defined by

$$dF_s(x)=\frac{\exp(sx)}{\phi(s)}dF(x), \text{ for each } s\in N.$$

Then its m.g.f. is given by

$$\phi_s(t)=\frac{\phi(s+t)}{\phi(s)}.$$

By condition 3), d.f. of the sum $\sum_{i=1}^{n} X_i$ is absolutely continuous for all $n\geq n_0$. An application of Lemma 1 gives us the inversion formula for all $n\leq n_0$:

Here we

(2) $$f_n(x)=\frac{n\exp(-n\tau x)}{2\pi}\phi^n(\tau)\int_{-\infty}^{\infty}\left\{\frac{\phi(\tau+it)}{\phi(\tau)}\right\}^n\exp(-itnx)dt \quad \text{for all } \tau\in N.$$

take into account that $f_n(x)=ng_n(nx)$, where $g_n(\cdot)$ is density of $\sum_{i=1}^{n} X_i$.

By condition 2), there exists a τ such that $\frac{\phi'(\tau)}{\phi(\tau)}=m$.

Since $\phi(z)$ is analytic in $Re(z)\in N$ and $\phi(\tau)>0$, we may form $\psi(z)=\log\phi(z)$ which is the principal branch of logarithm tending to $\log\phi(\tau)$ as $z\to\tau$ in some neighborhood of τ.

Since for all $\tau\in N$

$$0<\int\left(x-\frac{\phi'(\tau)}{\phi(\tau)}\right)^2 dF_\tau(x)$$
$$=\frac{\phi''(\tau)\phi(\tau)-\phi'(\tau)^2}{\phi(\tau)^2}=\left\{\frac{\phi'(\tau)}{\phi(\tau)}\right\}'=\{\log\phi(\tau)\}'',$$

$\{\log\phi(\tau)\}'$ is monotonic in N. Therefore we may invert $\frac{\phi'(\tau)}{\phi(\tau)}=x$.

Suppose $\frac{\phi'(\tau_n)}{\phi(\tau_n)}=x_n=m+\omega_n$ for all n.

Let $\varepsilon>0$ be a number smaller than the radius of the neighborhood within which log $\phi(z)$ is defined. To apply saddle point method, we write the last term in (2) as follows:

$$I_n=\int_{-\infty}^{\infty}\left\{\frac{\phi(\tau+it)}{\phi(\tau)}\right\}^n\exp(-intx)dt$$
$$=\int_{-\varepsilon}^{\varepsilon}\left\{\frac{\phi(\tau+it)}{\phi(\tau)}\right\}^n\exp(-intx)dt+R_n$$

where

$$R_n=\int_{|t|\geq\varepsilon}\left\{\frac{\phi(\tau+it)}{\phi(\tau)}\right\}^n\exp(-intx)dt$$

For $n\geq n_0$, $\phi^n(\tau+it)$ is the m.g.f. of the absolutely continuous d.f. and $\phi^n(z)$ is also analytic on the strip $Re(z)\in N$. We have by Lemma 2 $\phi(s+it)\to 0$, as $|t|\to\infty$ uniformly in s for every small neighborhood of τ which is contained in N.

Consequently there exists a positive number $\alpha(\varepsilon)$ such that for $|t|\geq\varepsilon$

$|\phi^{n_0}(s+it)|\leq e^{-\alpha}\phi^{n_0}(s)$ for all s in some neighborhood of τ.

Hence for $n\geq n_0$ we obtain

(3)
$$|R_n| \leq \int_{|t|\geq\varepsilon} \left|\frac{\phi(\tau+it)}{\phi(\tau)}\right|^{n-n_0} \left|\frac{\phi(\tau+it)}{\phi(\tau)}\right|^{n_0} dt$$
$$\leq \exp\{-\alpha(n-n_0)/n_0\} \int_{-\infty}^{\infty} \left|\frac{\phi(\tau+it)}{\phi(\tau)}\right| dt$$
$$\leq K\exp\{-\alpha(n-n_0)/n_0\} = o(n^{-m}) \text{ for all } m>0.$$

Next we estimate the principal part in I_n.

Define, for $t\in(-\varepsilon, \varepsilon)$,

$$G_n(t)=itx_n+\psi(\tau_n)-\psi(\tau_n+it)$$

and

$$G(t)=itm+\psi(\tau)-\psi(\tau+it).$$

Absolute continuity of d.f. for $n\geq n_0$ implies $ReG_n(t)>0$ for $|t|>0$. Also by expansion about zero we have

$$G_n(t)=c_nt^2+o(t^3) \text{ as } t\to 0.$$
$$\text{where } c_n=\frac{\psi''(\tau_n)}{2}>0.$$

Similarly we have $ReG(t)>0$ for $|t|>0$ and $G(t)=ct^2+o(t^3)$ as $t\to 0$, where $c=\frac{\psi''(\tau)}{2}=\frac{\sigma^2}{2}>0$.

Also $\omega_n=o(1)$ implies $c_n\to c$ as $n\to\infty$.

By expansion and analytic property of $\psi(\cdot)$, we have also

(4) $$\sup_n|G_n(t)-c_nt^2|=\sup_n|\psi'''(\tau_n+it_1)t^3/3!| \leq K|t|^3=0(t^3) \text{ as } t\to 0.$$

Hence the principal part of the integral I_n is

$$I_n'=\int_{-\varepsilon}^{\varepsilon}\exp\{-nG_n(t)\}\,dt$$
$$=\frac{1}{\sqrt{n}}\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\exp\{-nG_n(u/\sqrt{n})\}\,du.$$

Define $h_n(u)=\exp\{-nG_n(u/\sqrt{n})\}\,I_{(|u|<\varepsilon\sqrt{n})}(u)$

Clearly $I_{(|u|<\varepsilon\sqrt{n})}(u)(u)\to 1$ as $n\to\infty$ for all u.

And $nG_n(u/\sqrt{n})=n\{c_n(u/\sqrt{n})^2+0(u/\sqrt{n})^3\}=c_nu^2+n0(u^3/n^{3/2})$.

By (4), for fixed u, we have

$$nG_n(u/\sqrt{n})\to cu^2 \text{ as } n\to\infty.$$
$$\therefore\ h_n(u)\to h(u)=\exp(-cu^2) \text{ as } n\to\infty.$$

Now again by (4), we may choose $\varepsilon>0$ and n_1 such that for $n\geq n_1$,

$$nReG_n(u/\sqrt{n})\geq\frac{1}{2}c_nu^2 \text{ for all } |u|<\varepsilon\sqrt{n}.$$

Also $c_nu^2\to cu^2$ as $n\to\infty$ and $c>0$, $c_n>0$. Therefore there exists n_2 such that for $n\geq n_2$,

$$c_n u^2 \geq \frac{1}{2} c u^2 \text{ for all } u.$$

Thus for $n \geq \max(n_1, n_2)$,

$$nReG_n(u/\sqrt{n}) \geq \frac{1}{4} c u^2 \text{ for all } |u| < \varepsilon\sqrt{n}.$$

By dominated convergence, one has

$$\int_{-\infty}^{\infty} h_n(u)du \to \int_{-\infty}^{\infty} e^{-cu^2} du \text{ as } n \to \infty.$$

Consequently we obtain

$$I'_n \sim \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{-cu^2} du \text{ as } n \to \infty. \tag{5}$$

Substituting (3), (5) into (2) and taking into account that

$$\gamma(x) = \max_{s \in N} \{xs - \log\phi(s)\} = x\tau - \log\phi(\tau),$$

where τ is the unique $s \in N$ such that

$$\frac{\phi'(s)}{\phi(s)} = x \text{ holds,}$$

we have the result

$$f_n(x) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \exp\{-n\gamma(x)\}\{1 + o(1)\} \text{ as } n \to \infty.$$

Theorem is proved.

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