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Analysis of Modified Digital Costas Loop Part II : Performance in the Presence of Noise

(變形된 디지탈 Costas Loop에 關한 硏究 (II) 雜音이 있을 경우의 性能 解析)

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要 約

본 논문은 변형된 디지탈 Costas loop에 관한 논문으로서 제 1 부의 계속이다.

본 제 2 부 논문에서는 시스템에 잡음이 있을 경우 그의 성능을 해석하였다. 입력신호가 white Gaussian 잡음이 첨가되면 고려되는 DPLL의 noise process는 phase error detertor의 $\tan^{-1}(\cdot)$함수에 의해서 Rician이 됨을 보였다. 이 경우 Chapman-Kolmogorov 방정식을 수치적으로 풀므로써 1차와 2차 loop phase error의 steady state probability density함수, mean 및 variance를 얻었으며 이 결과를 컴퓨터 시뮬레이션에 의해서 입증하였다.

Abstract

This paper is a sequel of the Part I paper[1] on the modified digital Costas loop. In this Part II we analyze the performance of the system in the presence of noise. It is shown that, when the input signal is corrupted by additive white Gaussian noise, the noise process in the loop becomes Rician as a result of the $\tan^{-1}(\cdot)$ function of the phase error detector. Steady state probability density functions of phase errors of the first- and second-order loops have been obtained by solving the Chapman-Kolmogorov equation numerically. Also, the mean and variance of phase error in the steady state have been obtained analytically, and are compared with the results obtained by computer simulation.

I. Introduction

In part I of this paper[1] we introduced a new type of digital phase-lock loop (DPLL) called the modified digital Costas loop. This DPLL is composed of a 90° phase shifter, two samplers, a phase error detector with the $\tan^{-1}(\cdot)$ characteristic, a digital loop filter, and a digital clock. This loop is characterized by a linear difference equation which has a mod-2π feature. Unlike other DPLL's, this system has a unique property in that the phase error detector characteristic becomes linear as a result of insertion of $\tan^{-1}(\cdot)$ function in the loop. Assuming that the input signal was noiseless, we analyzed the first- and second-

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order loops by the phase plane technique. Locking ranges were obtained for different cases, and oscillation and false lock phenomena were observed for some initial conditions.

In the present study we investigate the performance of the same DPLL in the presence of noise based on the Chapman-Kolmogorov equation. Steady state probability density functions (pdf's) and variances of phase error are obtained for the first- and second-order loops. When the input signal is corrupted by band-passed additive white Gaussian noise, the noise process in the loop becomes Rician as a result of the $\tan^{-1}(\cdot)$ function of the phase error detector. This property will be discussed in detail.

Analyses of DPLL's in the presence of noise have been done by several researchers. A binary quantized DPLL that uses a sequential loop filter was studied by Cessna and Lavy.[2] Weinberg and Liu[3] analyzed a DPLL originally proposed by Gill and Gupta[4] using the Chapman-Kolmogorov equation. Also, Lindsey and Chie investigated the acquistion behavior of a first-order DPLL using the Chapman-Kolmogorov equation.[5] Recently, D'Andrea and Russo analyzed a first-order nonuniform multilevel quantized DPLL in the presence of phase- and frequency signal plus Gaussian noise.[6]

Following this introduction, we discuss the probability distribution of phase error and the difference equation of the system in Section II. In Section III we analyze the first-order modified digital Costas loop by using the Chapman-Kolomogorov equation. In this section we consider the steady state pdf, mean and variance of phase error. In Section IV we extend the analysis to the second-order loop. Finally, we draw conclusions in Section V.

II. Phase Error Stasistics and System Equation

The modified digital Costas loop is shown in Fig. 1. When the input signals to the samplers 1 and 2, x(t) and y(t), are sampled at the k_{th} sampling instant t(k), the sampled values are given respectively by

$$x(k) = \sqrt{2P_c}\ \sin\phi(k) + n(k) \qquad (1)$$

$$y(k) = \sqrt{2P_c}\ \cos\phi(k) + n'(k), \qquad (2)$$

where

$$\phi(k) \triangleq \omega_0 t(k) + \theta(k).$$

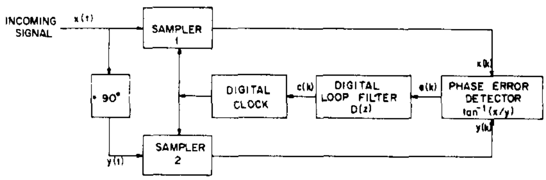

Fig. 1. Block diagram of modified digital Costas loop.

Here P_c is the power of information-bearing signal s(t); ω_0 is free-running frequency of the digital clock; $\theta(t)$ $[\Delta\omega \cdot t + \theta_0]$ is the phase process of s(t); $\Delta\omega$ is the initial frequency offset; and θ_0 is the initial phase offset. The noise process $\{n(k)\}$ is assumed to be a stationary and band-passed additive white Gaussian random process with zero-mean and variance of σ^2, and $\{n'(k)\}$ is the 90° phase-shifted noise process of n(k). Since the loop bandwidth of DPLL is much narrower than the input bandwidth, the noise process n(k) can be approximated by an independent and identically distributed Gaussian process.[5] The same is true for $\{n'(k)\}$.[7] Note that n(k) and n'(k) are mutually independent at a given time index k.[7]

The output of the phase error detector, e(k), is

$$e(k) = \tan^{-1}\left[\frac{x(k)}{y(k)}\right]. \qquad (3)$$

It can be shown that the pdf $p_e(e)$ of phase error signal e(k) is given by

$$P_e(e) = \frac{1}{2\pi}\cdot\exp\left(-\frac{\alpha^2}{2}\right) + \frac{\alpha\cos(e-\phi)}{2\pi}\exp\left[-\frac{1}{2}\alpha^2\sin^2(e-\phi)\right] \cdot \int_{-\infty}^{\alpha\cos(e-\phi)}\exp(-w^2/2)\,dw, \qquad (4)$$

where the parameter $\alpha(\triangleq \sqrt{2P_c}/\sigma)$ represents the input signal-to-noise ratio (SNR). In (4) we have omitted the time index k for convenience. Detailed derivation of (4) may be found in Appendix. It is to be noted that the above pdf of the phase error signal e(k) is Rician with its mean occurring at $e(k) = \phi(k)$. Since the phase error ranges from $-\pi$ to π, the pdf $P_e(e)$ must also vary from $e = -\pi$ to $e = \pi$. The phase error signal e(k) can be decomposed into the mean term $\phi(k)$ and the random variable $\eta(k)$ as

$$e(k) = \phi(k) + \eta(k). \tag{5}$$

In (5) one can interprete that $\phi(k)$ is the phase error signal at time t(k) when no noise is present, and $\eta(k)$ is the phase error noise process having Rician distribution with zero mean. The range of e(k) must be considered in the interval $(-\pi, \pi)$ in the mod-2π sense. Since n(k) and n′(k) are mutually independent white Gaussian noise processes, respectively, the phase error noise samples $\eta(k)$'s are also mutually independent for different k. The pdf $P_\eta(\eta)$ obtained from (4) and (5) is plotted in Fig. 2 for various input SNR's.

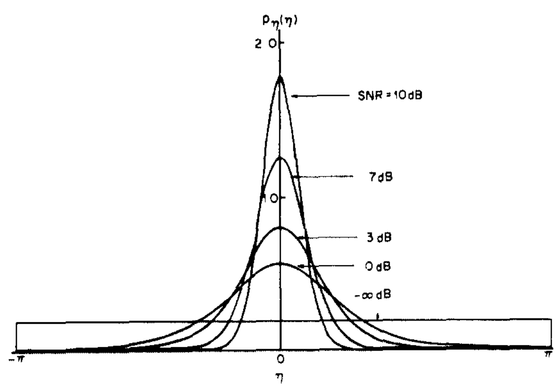

Fig. 2. Probability density function of phase error noise $\eta(k)$.

The system equation describing the loop behavior in the presence of noise can easily be obtained from the noiseless case[1] as

$$\phi(k+1) = \phi(k) - \omega \cdot D(z) \cdot [\phi(k)+\eta(k)] + 2\pi \frac{\Delta\omega}{\omega_0}, \tag{6}$$

where $\omega(=\omega_0+\Delta\omega)$ is the frequency of input signal, and D(z) is the transfer function of the digital loop filter. It is interesting to compare (6) with the corresponding equation of Gill and Gupta's DPLL[4] that is represented by a sinusoidal nonlinear difference equation as

$$\phi(k+1) = \phi(k) - \omega \cdot D(z)[\sqrt{2P_c}\sin\phi(k)+n(k)] + 2\pi \frac{\Delta\omega}{\omega_0}. \tag{7}$$

The system equation (6) of the loop under study is different from that of Gill and Gupta's in that $\sqrt{2P_c}\sin\phi(k)$ is substituted by $\phi(k)$ and Gaussian noise n(k) is replaced with Rician noise $\eta(k)$. Therefore, the behavior of the present loop is characterized by a mod-2π linear difference equation rather than a sinusoidal nonlinear equation. In the present system the phase error signal must be considered in the interval $(-\pi, \pi)$.

III. Analysis of First-Order Loop

1. Steady State pdf of Phase Error

When noise is present in the first-order loop, the system equation may be written from (6) as

$$\phi(k+1) = (1-\omega K)\,\phi(k) + 2\pi(\omega-1) - \omega K \cdot \eta(k), \tag{8}$$

where K is a constant. For convenience, from now on we shall assume the modulo 2π process without mentioning it. Obviously, one can regard this difference equation as a first-order Markov process. Accordingly, the pdf of $\phi(k)$ can be found from the following Chapman-Kolmogorov equation;

$$p_{k+1}(\phi/\phi_0) = \int_{-\pi}^{\pi} q_k(\phi/z)\, p_k(z/\phi_0)\, dz, \tag{9}$$

where ϕ_0 is initial phase error, $p_k(\phi/\phi_0)$ is the pdf of $\phi(k)$ given $\phi(0) = \phi_0$, $q_k(\phi/z)$ is the transition pdf of $\phi(k+1)$ given $\phi(k) = z$. As mentioned before, the noise term $\omega K\eta(k)$ has Rician distribution with zero mean, and its range is in the interval of mod-$2\pi[-\omega K\pi, +\omega K\pi]$ because phase error takes values in the interval $(-\pi, \pi)$. Therefore, $q_k(\phi/z)$ has Rician distribution of which the range is in the interval of mod-$2\pi[-\omega K\pi+(1-\omega K)z+2\pi(\omega-1), \omega K\pi+(1-\omega K)z+2\pi(\omega-1)]$, and the mean is located at $\phi = (1-\omega k)z + 2\pi(\omega-1)$. Accordingly, $q_k(\phi/z)$ is represented as

$$q_k(\phi/z) = \frac{1}{\omega K}\cdot\frac{1}{2\pi}\cdot\exp\left(-\frac{\alpha^2}{2}\right) + \frac{1}{\omega K}\cdot\frac{\cos\left(\frac{\phi-\phi_m}{\omega K}\right)}{2\pi}\cdot\exp\left(-\frac{\alpha^2\sin^2\left(\frac{\phi-\phi_m}{\omega K}\right)}{2}\right)\cdot\int_{-\infty}^{\alpha\cdot\cos\left(\frac{\phi-\phi_m}{\omega K}\right)}\exp\left(-\frac{w^2}{2}\right)dw, \tag{10}$$

where $\phi_m = (1-\omega K)z + 2\pi(\omega-1)$ and $\alpha = \frac{\sqrt{2P_c}}{\sigma}$.

From (9) the stady state pdf $p(\phi)$ satisfies the integral equation

$$p(\phi) = \int_{-\pi}^{\pi} q(\phi/z)\cdot p(z)\,dz, \tag{11}$$

where the time index k has been dropped, and $q(\phi/z)$ is given by (10). We obtain the steady state pdf $p(\phi)$ by solving (11) numerically. When we compute the steady state pdf numerically, we must consider the range of pdf's at the intermediate steps in mod-2π.

Simulation results along with calculated results are shown in Fig. 3 for various parameter values. Simulation was performed in the following way. First we segmented the interval $(-\pi, \pi)$ into L(=100) pieces. Second, we

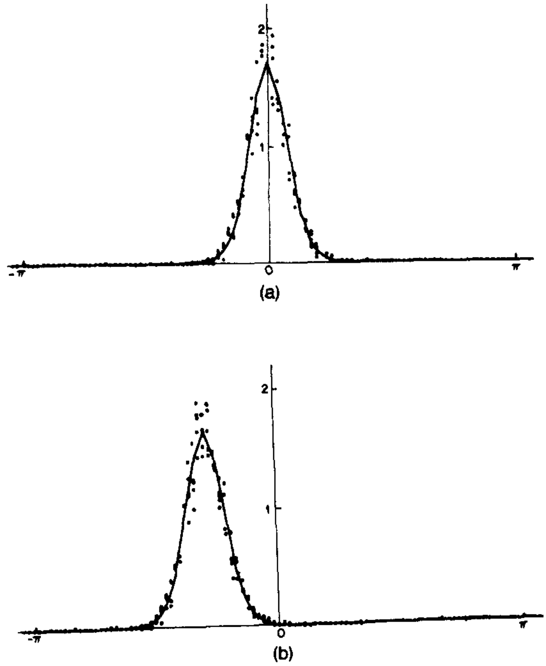

Fig. 3. Steady state pdf's of phase error of the first-order loop for various cases (SNR= 7dB). (Solid line is the numerical result and dotted points are the simulation results.)
(a) $\omega = 1$, $K = 0.7$
(b) $\omega = 0.9$, $K = 0.8$

generated Rician noise. Rician noise N_r can easily be generated from Gaussian noise N_g since it can be shown (see Appendix) that $N_r = \tan^{-1}\left(\frac{N_g}{1+N_g}\right)$ where the power of N_g can be set to a value one desires. Third, from(8) we computed M(=1000) phase error points, determined to which each point belongs among the L pieces, and then counted the number of phase error points in each segment. Lastly, we computed the probability density using the results from the third step. In the above process, the larger L and M, we can obtain the better results.

2. Steady State Mean and Variance

Let us now consider the steady state mean and variance of phase error.

Taking expectation of both sides of (8) in the steady state yields

$$E(\phi_{ss}) = (1-\omega K)E(\phi_{ss}) + 2\pi(\omega-1) - \omega K \cdot E(\eta). \tag{12}$$

Since $E(\eta) = 0$, we have

$$E(\phi_{ss}) = \frac{2\pi}{K}\left(1-\frac{1}{\omega}\right) \tag{13}$$

Note that $E(\phi_{ss})$ must lie in the interval $(-\pi, \pi)$. Similarly, squaring both sides of (8) and then taking expectation of each term in the steady state, we have

$$E(\phi^2_{ss}) = (1-\omega K)^2 E(\phi^2_{ss}) + \{2\pi(\omega-1)\}^2 + \omega^2 K^2 E(\eta^2) + 4\pi(\omega-1) \cdot (1-\omega K) E(\phi_{ss}), \tag{14}$$

and thus

$$E(\phi^2_{ss}) = E^2(\phi_{ss}) + \frac{\omega K}{2-\omega K} E(\eta^2). \tag{15}$$

Consequently, the steady state variance VAR (ϕ_{ss}) is given by

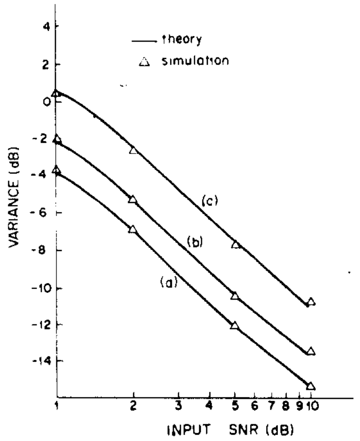

Fig. 4. Variances of phase error vs. input SNR of the first-order loop for various parameter values.
(a) $\omega = 1$, $K = 0.7$
(b) $\omega = 1.1$, $K = 0.8$
(c) $\omega = 1$, $K = 1.2$

$$VAR(\phi_{ss}) = \frac{\omega K}{2-\omega K} \cdot E(\eta^2). \tag{16}$$

It is to be noted that $VAR(\phi_{ss})$ is a function of the input SNR($= \frac{\alpha^2}{2}$) since $E(\eta^2)$ is a function of α. Variances of phase error together with simulation results are shown for various parameters in Fig. 4.

IV. Analysis of Second-Order Loop

We now analyze the second-order modified digital Costas loop in the presence of noise. As in Part I of the paper, we assume that the loop filter of the system takes the form of a proportional-plus-accumulation filter whose transfer function D(z) is given by

$$D(z) = a + \frac{b}{1-z^{-1}},$$

where a and b are constants.

1. Steady State Pdf of Phase Error

The system equation describing the loop in the present case is from (6)

$$\phi(k+2) = [2-(a+b)\omega]\phi(k+1) - (1-a\omega)\phi(k) - (a+b)\omega\eta(k+1) + a\omega\eta(k). \tag{17}$$

This form does not assure that the phase error process is a Markov process. Therefore, we write (17) in a set of first-order difference equations as

$$x_1(k+1) = [2-(a+b)\omega]\, x_1(k) - (1-a\omega)\, x_2(k) + a\omega\eta(k) \tag{18a}$$

$$x_2(k+1) = x_1(k) \tag{18b}$$

$$\phi(k) = -\left(1+\frac{b}{a}\right) x_1(k) + x_2(k). \tag{18c}$$

Then, from the above equations the phase error process can be regarded as a first-order two dimensional Markov process. Accordingly, to find the pdf of $\phi(k)$, we first obtain the joint pdf of $x_1(k)$ and $x_2(k)$ from the following Chapman-Kolmogorov equation;

$$P_{k+1}(x_1,x_2) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} q_k(x_1, x_2/z_1, z_2)\, p_k(z_1, z_2/x_{10}, x_{20})dz_1,\, dz_2, \qquad (19)$$

where $p_k(x_1, x_2/x_{10}, x_{20})$ is the joint pdf of $x_1(k)$ and $x_2(k)$ given $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$, and $q_k(x_1, x_2/z_1, z_2)$ is the transition joint pdf of $x_1(k+1)$ and $x_2(k+1)$ given $x_1(k) = z_1$ and $x_2(k) = z_2$. Then, from (18c), the pdf of $\phi(k)$ can be found. Note that the transition joint pdf $q_k(x_1, x_2/z_1, z2)$ is represented as the product of Rician distribution of random variable x_1 and a delta function $\delta(x_2-z_1)$ (i.e., distribution of random variable x_2). The range of x_1 is in the interval of mod-$2\pi[-a\pi\omega+[2-(a+b)\omega]z_1 - (1-a\omega)z_2,\ a\pi\omega+[2-(a+b)\omega]z_1-(1-a\omega)z_2]$ and its mean is located at $[2-(a+b)\omega]z_1-(1-a\omega)z_2$. Hence, $q_k(x_1, x_2/z_1, z_2)$ is represented as

$$q_k(x_1, x_2/z_1, z_2) = \Big[\frac{1}{a\omega}\cdot\frac{1}{2\pi}\exp\Big(-\frac{\alpha^2}{2}\Big) + \frac{1}{a\omega}\cdot\frac{\alpha\cos\big(\frac{x_1-x_{1m}}{a\omega}\big)}{2\pi}\cdot\exp\Big(-\frac{\alpha^2\sin^2\big(\frac{x_1-x_{1m}}{a\omega}\big)}{2}\Big)\cdot\int_{-\infty}^{\alpha\cos\left(\frac{x_1-x_{1m}}{a\omega}\right)}\exp\Big(-\frac{w^2}{2}\Big)dw\Big]\cdot\delta(x_2-z_1) \qquad (20)$$

where

$$x_{1m} = [2-(a+b)\omega]z_1 - (1-a\omega)z_2, \text{ and } \alpha = \frac{\sqrt{2P_c}}{\sigma}$$

Noting that the steady state joint pdf $p(x_1, x_2)$ satisfies the integral equation

$$p(x_1, x_2) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} q(x_1, x_2/z_1, z_2)\, p(z_1, z_2)\, dz_1 dz_2, \qquad (21)$$

one can obtain the steady state pdf $p(\phi)$ of phase error from (18c) and (21). Since it is difficult to solve (21) analytically, we obtain the steady state pdf $p(\phi)$ by solving (20) and (18c) numerically.

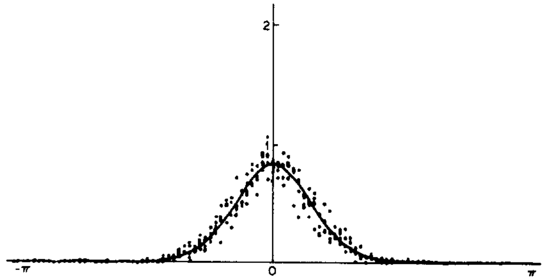

Fig. 5. Steady state pdf of phase error of the second-order loop (SNR = 7dB, ω = 1, a = b = 0.7)

In Fig. 5, simulation results that have been obtained with (17) are shown along with numerical results for different parameter values. The simulation procedure was the same as that used for the first-order loop.

2. Steady State Mean and Variance

We now consider the steady state mean and variance of phase error. Taking expectations of both sides of (18) in the steady state, we have

$$E(x_1) = [2-(a+b)\omega]\, E(x_1) - (1-a\omega)\, E(x_2) + a\omega\, E(\eta) \qquad (22a)$$

$$E(x_2) = E(x_1) \qquad (22b)$$

$$E(\phi_{ss}) = -\Big(1+\frac{b}{a}\Big) E(x_1) + E(x_2). \qquad (22c)$$

Clearly, $E(\phi_{ss})$ becomes zero since $E(\eta) = 0$.

Now, squaring both sides of (18) and then taking expectation of each term, we obtain in the steady state

$$E(x_1^2) = [2-(a+b)\omega]^2]\, E(x_1^2) + (1-a\omega)^2 E(x_2^2) + (a\omega)^2 E(\eta^2) - 2(1-a\omega)\cdot[2-(a+b)\omega]\, E(x_1\cdot x_2) \qquad (23a)$$

$$E(x_2^2) = E(x_1^2) \qquad (23b)$$

$$E(\phi^2_{ss}) = (1+\frac{b}{a})^2 E(x_1{}^2) + E(x_2{}^2)$$

$$- 2(1+\frac{b}{a})E(x_1 \cdot x_2) . \quad (23c)$$

Also, taking expectation of both sides of the product of (18a) and (18b) in the steady state, we have

$$E(x_1 \cdot x_2) = [2-(a+b)\omega]\, E(x_1{}^2)$$
$$- (1-a\omega)E(x_1 \cdot x_2) . \quad (24)$$

Note that $E(\phi_{ss}) = 0$. Hence, from (23) and (24), the steady state variance of the phase error, $\sigma^2_{\phi_{ss}}$, is given by

$$\sigma^2_{\phi_{ss}} = E(\phi^2_{ss}).$$

$$= \{[(a+b)^2\omega^2 + (a\omega)^2]\ (2-a\omega)-2a\omega(a+b)\ \omega[2-(a+b)\omega]\} \ / \ \{a\omega[(2-a\omega)^2 - - 2-(a+b)\omega^2]\} \cdot E(\eta^2)$$

$$(25)$$

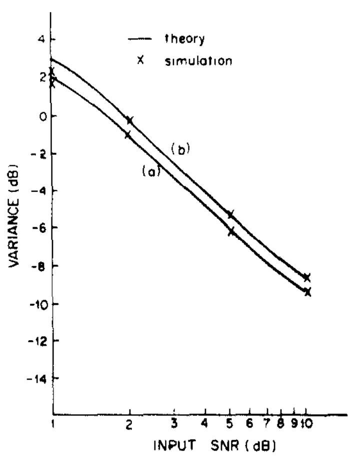

Fig. 6. Variances of phase error of the second-order loop.
(a) $\omega = 1$, $a = b = 0.7$
(b) $\omega = 1.1$, $a = b = 0.7$

In Fig. 6 variances of phase error in the steady state are shown along with simulation results for various values of the loop filter parameters. It is seen that the analysis agrees excellently with simulation, and that as in the case of APLL's, variance of phase error decreases as the input SNR increases.

V. Conclusions

We have studied the performance of the modified digital Costas loop in the presence of additive Gaussian noise. As a result of having the $\tan^{-1}(\cdot)$ function instead of a multiplier in the phase error detector, the loop is characterized by a linear difference equation and the probability distribution of phase error becomes Rician. We have obtained by solving the Chapman-Kolmogorov equation numerically the steady state probability distributions of phase error of the first- and second-order loops for different parameter values, and compared those with simulation results. Also, the steady state mean and variance of phase error have been obtained analytically. The results are in excellent agreement with simulation results.

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Appendix

Here we derive Eq. (4), the pdf $P_e(e)$ of phase error e(k). Since n(k) and n′(k) are stationary, mutually independent Gaussian noise processes, phase error signal e(k) given by (3) may be considered as a stationary noise process. For convenience, (1), (2) and (3) are rewritten;

$$x = \sqrt{2P_c}\ \sin\phi + n\ , \quad (1)$$

$$y = \sqrt{2P_c}\ \cos\phi + n'\ , \quad (2)$$

$$e = \tan^{-1}[x/y]\ , \quad (3)$$

where we have omitted the time index k. Note that n and n′ are mutually independent Gaussian random variables, and thus x and y are also mutually independent Gaussian random variables. The means and the variances of the random variables x and y are given, respectively, by

$$E[x] = \sqrt{2P_c}\ \sin\phi\ ,\ E[y] = \sqrt{2P_c}\ \cos\phi \tag{A·1a}$$

$$VAR[x] = VAR[y] = VAR[n] = \sigma^2. \tag{A·1b}$$

The joint pdf f(x,y) of x and y is given by

$$f(x,y) = \frac{1}{2\pi\sigma^2}\ \exp[-\frac{1}{2\sigma^2}\ (x-\sqrt{2P_c}\,(\sin\phi)^2 + (y-2\sqrt{2P_c}\ \cos\phi)^2\ ]. \tag{A·2}$$

Now, we wish to find the pdf of the random variable e given by (3). Let e and ϵ be written in the following forms;

$$x = \epsilon \sin e \tag{A·3a}$$

$$y = \epsilon \cos e\ . \tag{A·3b}$$

To find the pdf of e, the joint pdf of e and ϵ must be found first. For this purpose, we determine Jacobian $J(x,y;\ e,\epsilon)$ using (A·3) as

$$|J(x,y;e,\epsilon)| = \left\| \begin{matrix} \frac{\partial x}{\partial e} & \frac{\partial x}{\partial \epsilon} \\ \frac{\partial y}{\partial e} & \frac{\partial y}{\partial \epsilon} \end{matrix} \right\| = \epsilon. \tag{A·4}$$

Therefore, the joint pdf $g(e,\epsilon)$ of e and ϵ is given by

$$g(e,\epsilon) = \frac{1}{2\pi\sigma^2}\ \cdot\epsilon\cdot\exp[-\frac{1}{2\sigma^2}\left\{(\epsilon\cos e-\sqrt{2P_c}\cos\phi)^2 + (\epsilon\sin e-\sqrt{2P_c}\sin\phi)^2\right\}]$$

$$= \frac{1}{2\pi\sigma^2}\ \cdot\epsilon\cdot\exp[-\frac{P_c}{\sigma^2}\sin^2(e-\phi)] \cdot\exp[-\frac{1}{2\sigma^2}\left\{\epsilon-\sqrt{2P_c}\ \cos(e-\phi)\right\}^2]. \tag{A·5}$$

Note that, since ϵ represents a distance, it must always be equal to or greater than zero. Integrating $g(e,\epsilon)$ from zero to infinity with respect to ϵ, we obtain

$$P_e(e) = \int_0^\infty g(e,\epsilon)\ d\epsilon$$

$$= \frac{1}{2\pi\sigma^2}\ \exp[-\frac{P_c}{\sigma^2}\sin^2(e-\phi)]\cdot\int_0^\infty \epsilon \cdot\exp[-\frac{1}{2\sigma^2}\ \epsilon-\sqrt{2P_c}\ \cos(e-\epsilon)\ ^2]\ d\epsilon \tag{A·6}$$

If we change variables as

$$w \triangleq \frac{\epsilon - \sqrt{2P_c}\ \cos(e-\phi)}{\sigma}\ ,$$

then,

$$\epsilon\ = \sigma w + \sqrt{2P_c}\ \cos(e-\phi) \tag{A·7a}$$

$$d\epsilon = \sigma dw\ . \tag{A·7b}$$

Consequently, the lower limit of integration in (A·6) is changed to $-\sqrt{2P_c}\ \cos(e-\phi)/\sigma$. Hence, (A·6) may be written as

$$P_e(e) = \frac{1}{2\pi}\ \exp[-\frac{P_c}{\sigma^2}\sin^2(e-\phi)] \int_{-\frac{\sqrt{2P_c}\ \cos(e-\phi)}{\sigma}}^{\infty} [w+\frac{\sqrt{2P_c}}{\sigma}\cos(e-\phi)] \cdot\exp(-w^2/2)\ dw$$

$$= \frac{\exp(-P_c/\sigma^2)}{2\pi} + \frac{\sqrt{2P_c}\ \sigma}{2\pi}\ \cos(e-\phi) \cdot\exp[-\frac{P_c}{\sigma^2}\sin^2(e-\phi)] \int_{-\infty}^{\frac{\sqrt{2P_c}}{\sigma}\cos(e-\phi)} \exp(-w^2/2)\ dw. \tag{A·8}$$

Since $\alpha = \sqrt{2P_c}/\sigma$, the pdf $P_e(e)$ of the phase error signal e(k) is represented as follows;

$$P_e(e) = \frac{\exp(-\alpha^2/2)}{2\pi} + \frac{1}{2\pi}\,\alpha \cdot \cos(e-\phi)$$

$$\cdot \exp\left[\frac{-\alpha^2 \sin^2(e-\phi)}{2}\right]$$

$$\cdot \int_{-\infty}^{\alpha\cos(e-\phi)} \exp(-w^2/a)\, dw, \qquad (A\cdot 9)$$

which is Rician distribution given by (4).

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