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累積, 損傷, 模型의 特異한 境遇에 關한 研究

(On a Special Case of a Problem in the Cumulative Damage Model)

李 曉 成 *

ABSTRACT

An equipment is subject to shocks which occur randomly according to a Poisson process. The equipment fails when the total cumulative damage reaches a certain prescribed level. An optimal preventive replacement policy is presented when the level of cumulative damage cannot be checked. Renewal theory is used to obtain the results and a case example is presented.

I. Introduction

If the failure of an equipment during its operation is costly or dangerous, it is advantageous to replace the equipment before failure, i. e., use the preventive replacement policy. [1, 2]

In this paper, an equipment is considered which is exposed to shocks occurring randomly in time according to a Poisson process with parameter λ: The i^{th} shock causes a random amount X_i of damage, where $X_1, X_2, \cdots\cdots$ are independently distributed with common distribution F. The equipment fails when the total accumulated damage level reaches a specified threshold value W. The replacement policy of such a case was studied by T. Nakagawa when the cumulative damage level is monitored and the time span is infinite. And he obtained the optimal replacement damage level as a criterion of the preventive replacement policy. [3]

But If the cumulative damage level cannot be monitored or checking such a level entails too much cost, we cannot use the above policy. In such cases, we should use the number of shocks which have occurred to the equipment as the criterion of the preventive replacement policy. That is, we should replace the equipment at failure or replace just after a prescribed number of shocks have occurred, whichever comes first.

In this paper, under this policy, the optimal number of shocks before replacement shall be derived.

II. Notations and Assumptions

Notations

X_i $(i = 1, 2, \cdots)$; amount of damage caused by the i^{th} shock. r. v.

T_i $(i = 1, 2, \cdots)$; interarrival time between the succesive shocks i^{th} and $i+1^{th}$. r. v.

$f(.)$, $F(.)$; p. d. f. and c. d. f. of r. v. X_i

$g(.)$, $G(.)$; p. d. f. and c. d. f. of r. v. T_i

$F^k(.)$; the k-fold convolution of $F(.)$

W ; cumulative damage level at which eq·

* Sung Kyun Kwan University

uipment fails

c_f ; cost of the failure replacement

c_p ; cost of the preventive replacement

n ; number of shocks before replacement under this policy

C(n) ; expected cost per unit time if n is used as number of shocks before replacement

T(n) ; expected length of a replacement cycle if n is used as number of shocks before replacement

$P_p(n)$; portion of the preventive replacement if we use n as number of shocks before replacement.

$P_f(n)$; portion of the failure replacement if we use n as number of shocks before replacement

$N_p(t,n)$; number of preventive replacement during [o, t] if we use n as number of shocks before replacement

$N_f(t,n)$; number of failure replacement during [o, t] if we use n as number of shocks before replacement

Assumptions

1) Time span is infinite.

2) Shocks occur according to a Poisson process with parameter λ.

3) X_i and T_j are stochastically independent ($j \neq i$).

4) Replacement takes negligible time.

Ⅲ. Derivation of the Optimal Replacement Policy

If we use n as number of shocks before replacement, the expected cost per unit time in a replacement cycle becomes.

$$C(n)=\lim_{t\to\infty}\frac{c_f E[N_f(t,n)]+c_p E[N_p(t,n)]}{t}$$

$$=\lim_{t\to\infty}\frac{c_f[t/T(n)]P_f(n)+c_p[t/T(n)]P_p(n)}{t}$$

$$=\frac{c_f P_f(n)+c_p P_p(n)}{T(n)} \quad \cdots\cdots(1)$$

where

$$P_f(n)=Pr\{X_1\geq W\}+\sum_{j=2}^{n}P_r\{X_1+X_2+\cdots+X_{j-1}<W,\ X_1+X_2+\cdots+X_j\geq W\}$$

$$=1-F(W)+\sum_{j=2}^{n}\int_0^W[1-F(W-u)]f^{j-1}(u)\,du$$

$$=1-F(W)+\sum_{j=2}^{n}\{F^{j-1}(W)-F^j(W)\}$$

$$=1-F^n(W) \quad \cdots\cdots(2)$$

$$P_p(n)=1-P_f(n)=F^n(W) \quad \cdots\cdots(3)$$

Expected length of a replacement cycle becomes.

$$T(n)=\sum_{j=1}^{n}\int_0^\infty t\,Pr\{X_1+X_2+\cdots+X_{j-1}<W \text{ and } X_1+X_2+\cdots+X_j\geq W\}g^n(t)\,dt+\int_0^\infty F^n(W)\,tg^n(t)\,dt$$

$$=\sum_{j=1}^{n}Pr\{X_1+X_2+\cdots+X_{j-1}<W \text{ and } X_1+X_2+\cdots+X_j\geq W\}\int_0^\infty tg^n(t)\,dt+F^n(W)\int_0^\infty tg^n(t)\,dt$$

$$=\sum_{j=1}^{n}\{F^{j-1}(W)-F^j(W)\}\frac{j}{\lambda}+F^n(W)\frac{n}{\lambda}$$

$$=\frac{1}{\lambda}\sum_{j=1}^{n-1}F^j(W) \quad \cdots\cdots(4)$$

where $F^\circ(W)=1$

If we substitute (2), (3), (4) into (1), the expected cost per unit time becomes

$$C(n)=\frac{c_f[1-F^n(W)]+c_pF^n(W)}{\frac{1}{\lambda}\sum_{j=0}^{n-1}F^j(W)} \quad \cdots(5)$$

The function C(n) is a unimodal function of n, and the positive integer n which minimizes the equation (5) becomes the optimal number of shocks before replacement, which shall be denoted as n^*. And n^* can be obtained easily by solving the following difference inequality, $C(n-1) \geq C(n) \leq C(n+1)$, which can be solved readily by numerical method using computer.

Ⅳ. Comparison of the Two Models

If the cumulative damage level can be monitored, we should use the optimal replacement damage level as the best criterion of the replacement policy. Under that policy, an equipment is replaced when the total amount of damage exceeds a certain level k for the first time after each shock comes.

The following theorem derived in reference [3] gives the way for obtaining the optimal replacement damage level k^*. (In the theorem, $M(\cdot)$ denotes the renewal function of $F(\cdot)$).

– Theorem –

(i) If $M(W) > \frac{c_p}{c_f - c_p}$, then there exists a unique k^* $(0 < k^* < W)$ which satisfies the following equation

$$\int_{W-k}^{W} [1+M(W-u)]\, dF(u) = \frac{c_p}{c_f - c_p} \quad \cdots\cdots (6)$$

(ii) If $M(W) \leq \frac{c_p}{c_f - c_p}$, then $k^* = w$, i. e., use the failure replacement policy.

The expected cost per unit time for case (i) becomes

$$C(k^*) = \lambda (c_f - c_p)[1 - F(W-k^*)] \quad \cdots\cdots (7)$$

and for case (ii)

$$C(k^*) = \lambda c_f / [1 + M(W)] \quad \cdots\cdots (8)$$

Owing to the informations of the damage levels, $C(k^*)$ is always less than or equal to $C(n^*)$. But if the inspection cost per unit time is more than the cost of difference between $C(n^*)$ and $C(k^*)$, we should give up inspection activity and adopt our new policy.

Ⅴ. Case Example

Consider an equipment which fails only when the total wear amount of its certain component reaches 10 cm. And the wear of the component caused by each use of the equipment is assumed to obey the exponential distribution with mean 1 cm. The failure of the equipment during use is supposed to cost 1,000 dollars (which includes the replacement cost of the component) and the preventive replacement cost of the component to be 100 dollars. If the interarrival time of the successive uses of the equipment ooeys the exponential distribution with mean 0.5 week and if the wear amount of the component cannot be inspected by some reasons (cost, time, technical reason, etc.), what is the optimal number of uses before replacement?

If we put the above example in the context of our model, we shall get $W=10$, $\lambda=2$, $c_f=1{,}000$, $c_p=100$, $f(x)=\exp(-x)$. Then, with these informations, if we minimize the equation (5) with respect to n, the following result can be obtained by using computer.

Table 1. Result of the Case Example

C (n)	C (1)	C (2)	C (3)	C (4)	C (5)	C (6)
The Value of C(n)	200.082	100.452	68.341	54.697	50.669	53.884

So the optimal number of uses before replacement becomes 5 and the expected cost per week under optimal policy is 50.669 dollars.

If the wear level can be inspected, as the following inequality holds in the above example $M(W) = 10 > \frac{100}{1000-100} = \frac{c_p}{c_f - c_p}$, the optimal replacement wear level can be obtained by the equation(6). And after a little caculation, equation (6) becomes $k \exp[-(10-k)] = 1/9$. By solving that equation, we can get the optimal wear level k^*, which is 6.01cm.

By the equation (7), the expected cost per unit time is

$$C(k^*) = (c_f - c_p)[1 - F(W - k^*)] = 33.300$$
dollars.

So the expected cost of difference between $C(n^*)$ and $C(k^*)$ is 17.369 dollars. Consequently, if the average inspection cost per week entails less than 17.369 dollars, we should use the Nakagawa's policy, but otherwise, we should adopt our policy based on the number of shocks.

VI. Conclusion

In this paper, the optimal preventive replacement policy was presented when the failure of the equipment is subject to the cumulative damage process. This model can be applied to the specific cases when the cumulative damage level cannot be checked or checking such levels costs too much money or time.

If we meet the cases when we only know whether the equipment has failed or not, and not only the cumulative damage level but also the number of shocks occurred to the equipment cannot be checked, we should use the age replacement policy of Barlow and Proschan. [1]

REFERENCES

[1] Barlow, R. E. and Proschan, F., Mathematical Theory of Reliability, John wiley and Sons, Inc., New York, 1965.

[2] Barlow, R. E. and Proschan, F., Statistical Theory of Reliability and Life Testing Probability Models, Holt, Rinehart and Winston, Inc., 1975.

[3] Nakagawa, T., "On a Replacement Problem of a Cumulative Damage Model," Opl Res. Q., Vol. 27, 4, pp. 895 – 900, 1976.