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CONSTRAINED OPTIMIZATION PROBLEMS IN TRIPLE RESPONSE SYSTEMS

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I. INTRODUCTION

Often in experimental work, the researcher is called upon to seek the conditions of experimentation which are most desiable, depending upon some preseleted criterion. Much has been written concerning the exploration of response surfaces. Basically, a polynomial type response function is used to explore the relationship between a response variable and k independent variables which is given by

$$\eta = g(x_1, x_2, \cdots\cdots, x_k)$$

in some region of interest. The most frequently fitted response function is the quadratic model which gives rise to a fitted response function of the form

$$\hat{y} = b_0 + \Sigma_j b_j x_j + \Sigma_j \Sigma_m b_{jm} x_j x_m + \Sigma_j b_{jj} x_j^2 \quad (1.1)$$
$$j < m$$

The fitted second order function in (1.1) is given in matrix notation by

$$\hat{y} = b_0 + x' b + x' Bx$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_k \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_k \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12}/2 & \cdots & b_{1k}/2 \\ & b_{22} & \cdots & b_{2k}/2 \\ & & & \vdots \\ \text{sym} & & & b_{ku} \end{bmatrix}$$

Here, X is a vector of independent or design variables and $\hat{y}$ is the estimated response. The elements in b and B represent least squares estimators.

The total exploration following the estimation of (1.2) often involves finding the stationary point. This stationary point will be used to aid in describing the response surface system. The derivative of $\hat{y}$ with respect to vector x, equated to O, gives

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}[x' b + x' Bx] \quad x$$
$$= b + x' Bx = 0$$

Solving for X we have the stationary point Xo given by

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$$Xo = -B^{-1} b/2$$

If the stationary point falls inside the domain of approximation of the second order response function, a Cannonical analysis can be conducted to determine the nature of the stationary point and the properties of the system in the region of the experimental design. Otherwise the Rideg analysis can be conducted. Discussion of these procedures are given in Box [1] and Myers [2].

Quite often the researcher is confronted with the need to simultaneously optimize several response variables. It is not unusual in this situation to obtain a solution X, which is optimal for one response and far from optimal or even physically impractical for the others. The task is then to arrive at some comprise conditions involving the responses. The method which super imposes response contours to arrive at suitable operating conditions cannot be well applied when the number of independent variables exceeds three. Hence, the purpose of this paper is to present the theory and develop an algorithm associated with the exploration of a triple response surface system.

The approach is to find condition on a set of independent or design variables which maximize (or minimize) a primary response function subject to the condition that constrained response functions take on some specified or desiable values. A method is outlined whereby a user can generate simple two dimensional plots to determine the conditions of optimum primary response regardless of the number of independent variables in the system. The approach described in this paper is similar to that of Myer [3] which presents an algorithm associated with the exploration of a dual response surface system. In some point of view, the approach of Myers is the generalization of that of Draper [4].

II. THE TRIPLE RESPONSE PROBLEM

Let us suppose that the experimenter has a primary response with fitted response function given by

$$\hat{y}_p = b_0 + x'b + x'Bx \qquad (2.1)$$

and what we shall refer to as two constraint responses with response functions given by

$$\hat{y}s = co + x'c + x'Cx \qquad (2.2)$$

$$\hat{y}q = do + x'd + x'Dx \qquad (2.3)$$

The expression in (2.2) and (2.3) may have been obtained from the same experiment. When the constraint response is the cost variable in a yield-cost study, the coefficients in (2.2) and (2.3) need not be random variables. The solution proposed and discussed in the sequel is to find the conditions on x which optimize $\hat{y}p$ subject to $\hat{y}_S = k_1$ and $y_q = k_2$, where k_1 and k_2 are some desiable or acceptable values of the constraint responses.

Using Lagrangian multipliers, we thus consider

$$L = b_0 + x'b + x'Bx - \mu(c_0 + x'c + x'Cx - k_1) - r(d_o + x'd + x'Dx - K_2)$$

and require solutions for x in the set of equations

$$\frac{\partial L}{\partial x} = 0$$

This results in the following :

$$(B - \mu C - \gamma D)\, x = \tfrac{1}{2}(\mu c + \gamma d - b) \qquad (2.4)$$

It is important at this point to study the nature of the stationary point generated by equation (2.4). We begin by considering the matrix M (x), of second partial derivatives, the (i, j) element of which is $(\partial^2 L/\partial x i \partial x j)$ (i, j = 1, 2, ⋯K).
It follows that

$$M(x) = 2(B - \mu C - \gamma D) \qquad (2.5)$$

Let's define $\hat{y}_{p,\ i}$ $\hat{y}_{S\ i}$ and $\hat{y}_{q,\ i}$ as following:

$$\hat{y}_{p,i} = b_0 + x_i'b + x_i'Bx_i$$
$$\hat{y}_{s,i} = c_0 + x_i'c + x_i'Cx_i$$
$$\hat{y}_{q,i} = d_0 + x_i'd + x_i'Dx_i$$

Theorem 2.1 : Let x_1 and x_2 be distinct solutions to equation (2.4) with some fixed value μ using γ_1 and γ_2 respectively. Let $\hat{y}_{s,1} = \hat{y}_{s,2}$ and $\hat{y}_{q,1} = \hat{y}_{q,2}$. If the matrix $(B - \mu C - \gamma_1 D)$ is negative definite, then $\hat{y}_{p,1} > \hat{y}_{p,2}$. If the matrix $(B - \mu C - \gamma_1 D)$ is positve definite, then $\hat{y}_{p,2} > \hat{y}_{p,1}$. In addition, if $(B - \mu C - \gamma_1 D)$ is negative definite, $(B - \mu C - \gamma_2 D)$ can not be negative definite.

Proof: If x_1 and x_2 give rise to the same value of the $\hat{y}_q$, then

$$d_0 + x_1'd + x_1'Dx_1 = d_0 + x_2'd + x_2'DX_2 \qquad (2 \cdot 6)$$

Consider now $\hat{y}_{p\,1} - \hat{y}_{p,2}$. We can write

$$\hat{y}_{p,1} - \hat{y}_{p,2} = x_1'Bx_1 - x_2'Bx_2 + (x_1' - x_2')b \qquad (2 \cdot 7)$$

By adding and substracting $(\mu x_2'Cx_2 + \gamma_1 x_2'Dx_2)$, we obtain

$$\hat{y}_{p,1} - \hat{y}_{p,2}$$
$$= x_1'Bx_1 - x_2'(B - \mu C - \gamma_1 D)x_2 - \mu x_2'Cx_2 - \gamma_1 x_2'Dx_2$$
$$+ (x_1' - x_2')b \qquad (2 \cdot 8)$$

From (2.4) with $\gamma = \gamma_1$ and $x = x_1$, we have

$$x_1'Bx_1 = \mu x_1'Cx_1 + \gamma_1 x_1'Dx_1 + \frac{1}{2}\mu x_1'c + \frac{1}{2}\gamma_1 x_1'd - \frac{1}{2}x_1'b$$

which from (2.3) becomes

$$x_1'Bx_1 = \gamma_1 y_{q,1} - \gamma_1 d_0 - \gamma_1 x_1'd + \mu x_1'Cx_1 + \frac{1}{2}\mu x_1'c + \frac{1}{2}\gamma_1 x_1'd$$
$$- \frac{1}{2}x_1'b \qquad (2 \cdot 9)$$

From (2.3) we also have

$$\gamma_1 x_2'dx_2 = \gamma_1 y_{q,2} - \gamma_1 d_0 - \gamma_1 x_2'd. \qquad (2 \cdot 10)$$

Hence, from (2.9) and (2.10) it follows that

$$x_1'Bx_1 - \gamma_1 x_2'Dx_2$$

$$= -\frac{1}{2}r_1 x_1'd + \gamma_1 x_2'd - \frac{1}{2}x_1'b + \mu x_1'Cx_1 + \frac{1}{2}\mu x_1'c$$

Thus, (2.8) becomes $y_{p,1} - y_{p,2}$

$$= -(x_1 - x_2)'(B - \mu C - \gamma_1 D)(x_1 - x_2) + \mu(\hat{y}_{s,1} - \hat{y}_{s,2})$$
$$= -(x_1 - x_2)'(B - \mu C - \gamma_1 D)(x_1 - x_2) \qquad (2 \cdot 11)$$

Which is postivie if $(B - \mu C - r_1 D)$ is a negative definite matrix and negative if $(B - \mu C - r_1 D)$ is a positive definite matrix. Equation (2. 11) also indicates

$$\hat{y}_{p,1} - \hat{y}_{p,2} = (x_1 - x_2)'(B - \mu C - \gamma_2 D)(x_1 - x_2)$$

which implies that while $(B - \mu C - r_1 D)$ is negative definite, $(B - \mu C - \gamma_2 D)$ can not be negative definite.

Theorem 2.1 indicates that in the quest for values of x which yield constrained maxima (minima) we can limit ourselves to values of γ which make $(B - \mu C - \gamma D)$ negative definite (positive definite) with some fixed value μ assuming that such values exist. It shall be demonstrated that this working region in γ does often exist and that its location depends on the nature of matrix B, C and D.

2.1 Positive definite case

Suppose that D is a positive definite matrix. Consider the quadratic form with matrix given by M(x), i.e., $q = W'(B - \mu C - \gamma D)W$ where μ is some fixed value. Since D is symmetric positive definite, there exists a non-singular matrix S (Rao [5] such that

$$S'(B - \mu C)S = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k)$$

and $S'DS = I_k$. Performing transformation $W' = V'S$

we have

$$q = V' \text{ diag } (\lambda_1 - r, \lambda_2 - r, \cdots, \lambda_k - r) \qquad (2.12)$$

The λ's are mearly the eigenvalues of the real symmetric matrix

$$T = D_2(-1/2)Q'(B-\mu C)Q\ D_2(-1/2) \quad (2\cdot13)$$

Here Q is the orthogonal matrix for which

$$Q'DQ = D_2 \quad (2.14)$$

and D_2 is the diagonal matrix containing the eigenvalues of D. We use the notation $D_2^{(-1/2)}$ to denote a diagonal matrix containing the reciprocals of the square roots of the eignevalues of D. From equation (2. 12), it is clear that we can insure a negative definite M(x) if $\gamma > \lambda_k$ (positive definite if $\gamma < \lambda_1$) where $\lambda_1, \lambda_2 \ldots, \lambda_k$ are the eigenvalues of the matrix T arranged in ascending order. In what follows, it becomes apparent that this indeed defines the working region for γ when μ is fixed. In fact, any $\gamma_1 > \lambda_k$ yields x_i which gives rise to an absolute maximum $\hat{y}_{p,i}$ (absolute minimum for $\gamma_i < \lambda_1$) conditional on being on two response surface given by $\hat{y}_{s,i}$ and $\hat{y}_{q,i}$. It turns out that by choosing some fixed μ and r values in this region on generates x's which give desiable value of $\hat{y}_s$ and $\hat{y}_q$.

Theorem 2.2 : Let x be a solution to (2. 4) where D is positve definite and μ is fixed. Then $\frac{\partial^2 \hat{y}_q}{\partial \gamma^2} \geq 0$ with the equality holding only in the limit as μ approaches $\pm\infty$

Proof: Differentiating both side of equation (2. 3) and (2. 4) with respect to r yields

$$\frac{\partial \hat{y}_q}{\partial \gamma} = d'\frac{\partial x}{\partial \gamma} + 2x'D\frac{\partial x}{\partial \gamma} \quad (2\cdot15)$$

and

$$(B-\mu C-\gamma D)\frac{\partial x}{\partial \gamma} = \frac{1}{2}d + Dx. \quad (2\cdot16)$$

Upon taking the second partial in (2. 15) and (2. 16) with respect to γ, one can write

$$\frac{\partial^2 \hat{y}_q}{\partial \gamma^2} = d'\frac{\partial^2 x}{\partial \gamma^2} + 2\left[x'D\frac{\partial^2 x}{\partial \gamma^2} + \frac{\partial x'}{\partial \gamma}D\frac{\partial x}{\partial \gamma}\right] \quad (2\cdot17)$$

$$(B-\mu C-\gamma D)\frac{\partial^2 x}{\partial \gamma^2} = 2D\frac{\partial x}{\partial \gamma} \quad (2\cdot18)$$

Upon premultiplying (2. 16) by ($\partial^2 x'/\partial r^2$) and (2. 18) by ($\partial x'/\partial r$) and subtracting the resulting eauations we find that

$$\frac{1}{2}d'\frac{\partial^2 x}{\partial \gamma^2} = 2\frac{\partial x'}{\partial \gamma}D\frac{\partial x}{\partial \gamma} - x'D\frac{\partial^2 x}{\partial \gamma^2} \quad 2\cdot19)$$

Substituting the expression for $\left(d'\frac{\partial^2 x}{\partial \gamma^2}\right)$ from (2. 19) into (2. 17) results in

$$\frac{\partial^2 y^q}{\partial \gamma^2} = 6\frac{\partial x'}{\partial \gamma}D\frac{\partial x}{\partial \gamma^2}$$

which, of course, is greater than zero except when $(\partial x/\partial \gamma) = 0$. From (2. 4) and (2. 16) $(\partial x/\partial \gamma) = 0$ only in the limite as γ approaches either plus or ninus infinity.

The theorem 2. 2 is useful in obtaining an understanding of the relationship between the Lagrangian multiplier γ and $\hat{y}_q$ when μ is fixed The relationship between γ and $\hat{y}_q$ is of the form illustrated in Figure (2. 1) In the figure, $\hat{y}_{o,q}$ is the value of the estimated constraint response function at its stationary point. The existence of the asymptotes is easily seen since from (2. 4).

$$\lim_{r\to\infty} x = -D^{-1}d/2 = x_{q,0} \quad \text{where } \mu \text{ is fixed.}$$

As γ approaches λ_i (i = 1, 2, . . , k), $\hat{y}_q$ approaches infinity since

$$|B-\mu C-\lambda_i D| = 0 \ \ (i=1, 2, \cdots, k)$$

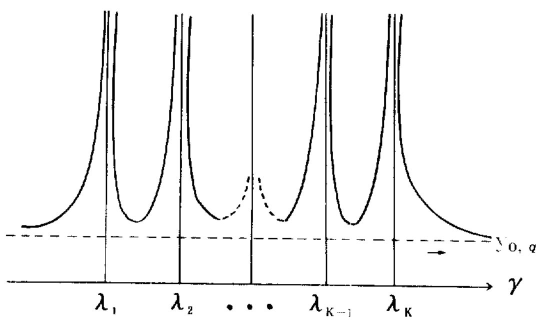

Figure 2.1 – Plot of $\hat{y}_q$ against γ with fixed μ.

Theorem (2. 1) and (2. 2) indicate the working region for γ resulting in a maximization of $\hat{y}_p$, subject to specific values of $\hat{y}_s$ and $\hat{y}_q$ is $\gamma > \lambda_k$ and $\gamma < \lambda_1$ for minimization. Hence, the procedure of determing operating conditions is

Step 1. Fix the some value of μ.

Step 2. Find the working region of γ.

Step 3. Evaluate $\hat{y}_p$, $\hat{y}_s$ and $\hat{y}_q$ with fixed μ and various γ in the working region.

Step 4. If the constraints are not satisfied, choose another μ and go to step 2. Otherwise stop.

Note that in this procedure whether C is definite or not does not matter. Numerical example of this procedure is given in the section 5. 1.

2.2 Negative Definite Case

Suppose that D is negative definite and C is negative definite or indefinite. In this case, in order to render M(x) negative definite, and thus find x values from (2. 4) which maximize $\hat{y}_p$ subject to constraints on $\hat{y}_s$ and $\hat{y}_q$, we are led to choosing values of γ with fixed μ which are smaller than the smallest elgenvalue of T. On the other hand, if our desire is to minimize $\hat{y}_p$, we find conditions by choosing γ larger than the largest eigenvalue of T.

Theorem 2. 3 : Let x be a solution to (2. 4) where D is negative definite and μ is fixed. Then $\frac{\partial^2 \hat{y}_q}{\partial r^2} \leq 0$ with the equality holding in the limit as r approaches $+\infty$.

Proof: The proof is similar to that of Theorem 2. 2.

Hence, the procedure of determing operating condition is similar to that of the positive definite case except step 2.

In case C and D are indefinite, it is impossible to use this approach, because we can not find a working region for γ which makes M(x) definite.

III. SOME SPECIAL CASES

Let us suppose that $\hat{y}_s$ and $\hat{y}_q$ are of the linear form given by

$$\hat{y}_p = b_0 + x'b + x'Bx$$
$$\hat{y}_s = c_0 + x'c$$
$$\hat{y}_q = d_0 + x'd$$

then it is the quadratic programming. Hence, it can be solved by Wolfe's Algorithm or Hildreth's Algorithm (see [6] [7]).

Let us suppose that c is zero matrix.

$$\hat{y}_p = b_0 + x'b + x'Bx \qquad (3\ \ 1)$$
$$\hat{y}_s = C_0 + x'C \qquad (3\ \ 2)$$
$$\hat{y}_q = d_0 + x'd + x'Dx \qquad (3\ \ 3)$$

This solution proposed is to find the conditions of x which optimize $\hat{y}_p$ subject to linear form $\hat{y}_s = k_1$ and Quadratic form $\hat{y}_q = k_2$, where k_1 and k_2 are some desiable or acceptable values of the constraint responses.

Using Lagrangian multipliers, we thus consider

$$L = b_0 + b'x + x'Bx - \mu(c_0 + x'c - k_1) - \gamma(d_0 + x'd + x'Dx - k_2)$$

and require solutions for x in the set of equations

$$\frac{\partial L}{\partial x} = 0$$

This results in the following:

$$(B - \gamma D)\, x = \frac{1}{2}(\mu c + \gamma d - b) \qquad (3 \cdot 4)$$

Consider the matrix, M(x) of second partial derivatives, the (i, j) element of which is $(\partial^2 L/\partial x_i \partial x_j)$ (i, j=1, 2, ⋯K)

If follows that

$$M(x) = 2(B - rD).$$

By similar idea in the previous section, we can find the working region for γ (see table 3. 1) resulting in an optimum of $\hat{y}_p$, subject to specific values of $\hat{y}_s$ and $\hat{y}_q$.

	Dposive definite	Dnegative definite	Dindefinite
maximize $\hat{y}_p$	$\gamma > \lambda_k$	$\gamma < \lambda_1$	if B is negative definite, $-\frac{1}{\lambda_k} < \gamma < -\frac{1}{\lambda_1}$
minimize $\hat{y}^p$	$\gamma < \lambda_1$	$\gamma > \lambda_K$	if B is positive definite, $\frac{1}{\lambda_1} < \gamma < \frac{1}{\lambda_K}$
	$\lambda_1, \lambda_2, \cdots, \lambda_K$ are eigenvalues of T arranged d in ascending order.		$\lambda_1, \lambda_2, \cdots, \lambda_K$ are eigenvalues of T^* arranged in ascending order.

Table 3·1. -The working region for γ.

i) $T = D_2^{(-\frac{1}{2})} Q'BQ D_2^{(-\frac{1}{2})}$

Here, Q is the orthoqonal matrix for which $Q'DO = D_2$ and D_2 is the diagonal matrix containing the eigenvalues of D. We use the notation $D_2^{(-\frac{1}{2})}$ to denote a diagonal matrix containing the reciprocals of the square roots of the eigenvalues of D.

ii) $T^* = D_1^{(-\frac{1}{2})} p'DP D_1^{(-\frac{1}{2})}$

Here, P is the orthogonal matrix for which $P'[-B]P = D_1$ and D_1 is the diagonal matrix containing the eigenvalues of $-B$ and $D_1^{(-\frac{1}{2})}$ contains reciprocals of the square roots of eigenvalues of $-B$.

Hence, the procedure of determining the operating condition is

Step 1. Find the workding region for γ.
Step 2. Take a γ in the working region and find μ that satisfies $\hat{y}_s = k_1$.

Note that μ is uniquely determined by (3. 4) and (3. 2) if such value exists.

Step 3. Evaluate $\hat{y}_p$ and $\hat{y}_q$ using γ and μ.
Step 4. If the constraint is not satisfied, go to step 2. Otherwise stop.

Numerical example of this procedure is given in the section 5. 2.

IV. NUMERICAL EXAMPLES

5.1 Definite Case

Consider a triple response surface problem where $\hat{y}_p$, $\hat{y}_s$ $\hat{y}_q$ depend on three independent variables x_1, x_2 and x_3. The region of the experiment on each variable is given

$$-1 \leq x_i \leq 1. \quad (i = 1, 2, 3)$$

The following three response functions were obtained from a set of experimental data.

$$\hat{y}_p = 65.39 + 9.24x_1 + 6.36x_2 + 5.22x_3 - 7.23x_1^2 - 7.76x_2^2 - 13.11x_3^2 - 13.68x_1x_2 - 18.92x_1x_3 - 15.46x_2x_3$$

$$\hat{y}_s = 56.42 + 4.65x_1 + 8.39x_2 + 2.56x_3 - 5.23x_1^2 - 4.37x_2^2 - 11.11x_3^2 - 13.68x_1x_2 - 18.92x_1x_3 - 15.52x_2x_3$$

$$\hat{y}_q = 59.37 + 2.53x_1 + 2.47x_2 + 5.62x_3 + 5.25x_1^2 + 5.62x_2^2 + 4.22x_3^2 + 8.74x_1x_2 + 2.32x_1x_3 + 3.78x_2x_3$$

The goal of the investigation is to determine operating condition which maximizes $\hat{y}_p$ but also we require $62 \leq \hat{y}_s \leq 64$ and $60 \leq \hat{y}_q \leq 62$. Recall that the matrix T is given by $T = D_2^{(-\frac{1}{2})} Q'(B-\mu C)QD_2^{(-\frac{1}{2})}$.

$$D_2^{(-\frac{1}{2})} = \begin{bmatrix} 0.3078 & 0 & 0 \\ 0 & 0.5302 & 0 \\ 0 & 0 & 0.0105 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.6428 & -0.3497 & 0.6816 \\ 0.6938 & -0.1115 & -0.7115 \\ 0.3248 & 0.9302 & 0.1710 \end{bmatrix}$$

i) $\mu=2.0$

$$T=\begin{bmatrix}0.5639 & 1.5286 & 1.1688\\ 1.5286 & 0.2733 & 0.9651\\ 1.1688 & 0.9651 & -4.2438\end{bmatrix}$$

with eigenvalues of T being (2.8960, –0.7346, –4.5684).

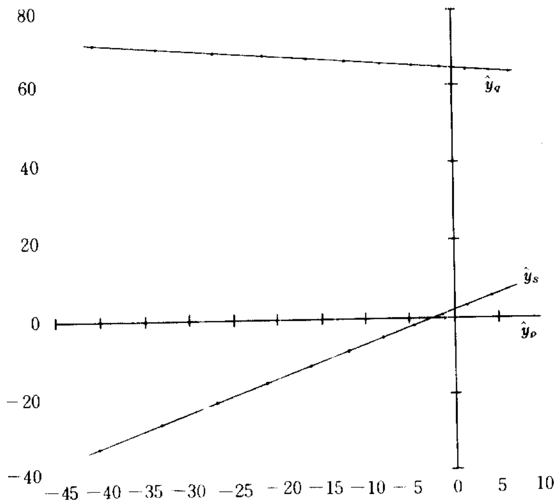

Figure 5.1

ii) $\mu=1.0$

$$T=\begin{bmatrix}-0.2350 & 0.0593 & 0.1987\\ 0.0593 & -0.5915 & -0.2123\\ 0.1987 & -0.2123 & -2.8649\end{bmatrix}$$

with eigen values of T being (–0.2148, –0.5765, –2.90)

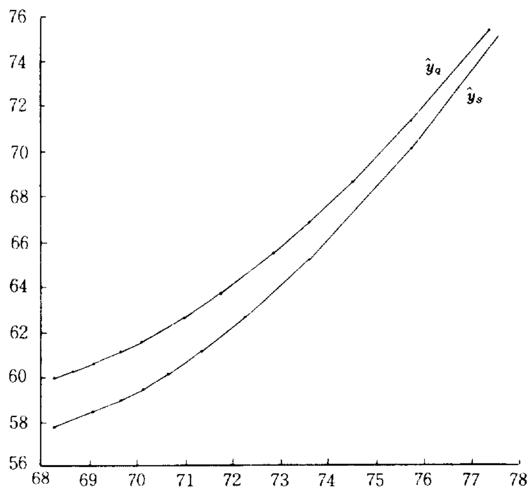

Figure 5.2

iii) $\mu=0.0$

$$T=\begin{bmatrix}-2.0339 & -1.4099 & -0.7713\\ -1.4099 & -1.4564 & -1.3896\\ -0.7713 & -1.3896 & -1.4859\end{bmatrix}$$

with eigen values of T being (0.0802 –0.9946, –4.0618)

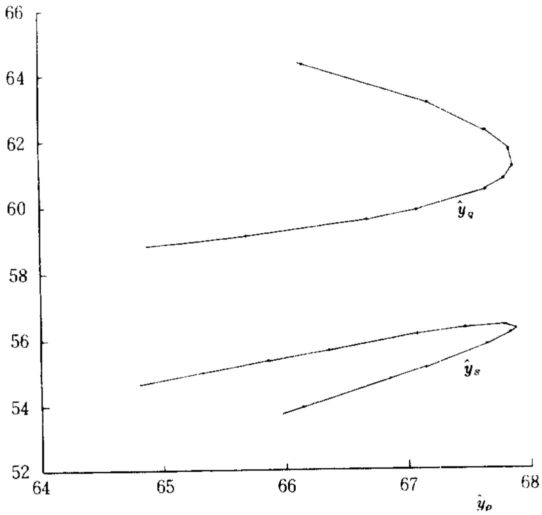

Figure 5.3

iv) $\mu=-1.0$

$$T=\begin{bmatrix}-3.8327 & -2.8792 & -1.7414\\ -2.8792 & -2.3213 & -2.5070\\ -1.7414 & -2.5670 & -0.1070\end{bmatrix}$$

with eigen values of T being (1.5862, –0.5615, –7.2857).

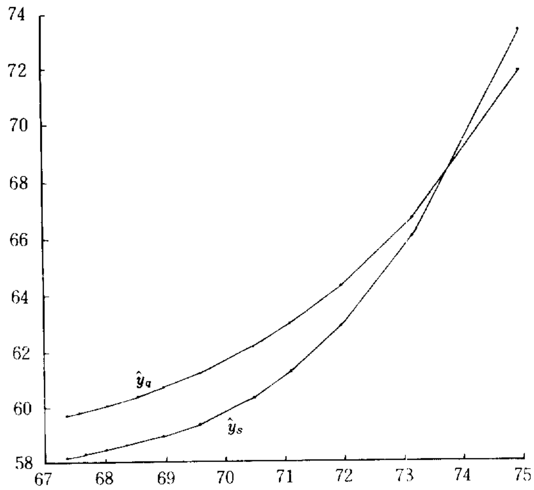

Figure 5.4

v) $\mu = -2.0$

$$T = \begin{bmatrix} -5.6316 & -4.3484 & -2.7115 \\ -4.3484 & -3.1861 & -3.7444 \\ -2.7115 & -3.7444 & 1.2720 \end{bmatrix}$$

with eigen values of T being (3.4079, −0.3560, −10.5975).

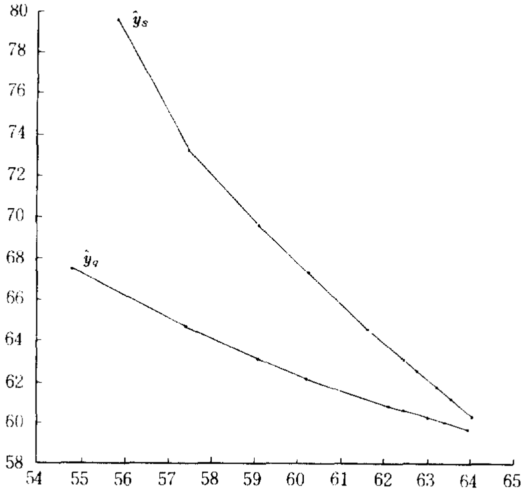

Figure 5.5

Note that the method to determine μ is trial-and-error method. After above procedure has been done with various values of μ. Figure 5. 5 gives the values of the estimated response, $\hat{y}_p$ = 63.01, $\hat{y}_s$ = 62.15 and $\hat{y}_q$ = 60.30 at the co-ordinates, x_1 = −0.47, x_2 = 0.86 and x_3 = −0.31.

5.2 Some Special Case

Consider a following response surface problem where $\hat{y}_p$, $\hat{y}_q$ and $\hat{y}_s$ depend on three independent variables x_1, x_2 and x_3. The region of the experiment on each variable is given by

$$0 \leq x_1 \leq 10, \quad -10 \leq x_2 \leq 10, \quad -1.0 \leq x_3 \leq 1.0$$

The following three response functions were obtained from a set of experimental data.

$$\hat{y}_p = 191.39 + 9.24x_1 + 6.39x_2 + 5.22x_3 - 7.23x_1^2 - 7.76x_2^2 - 13.11x_3^2 - 13.68x_1x_2 - 18.92x_1x_3 - 15.46x_2x_3$$

$$\hat{y}_s = 254.42 + 4.56x_1 + 8.93x_2 + 2.65x_3$$

$$\hat{y}_q = 56.42 + 4.65x_1 + 8.39x_2 + 2.65x_3 + 5.25x_1^2 + 5.62x_2^2 + 4.22x_3^2 + 8.74x_1x_2 + 2.32x_1x_3 + 3.78x_2x_3$$

The goal of the investigation is to determine operating condition which maximizes $\hat{y}_p$ but also we require

$$204 \leq \hat{y}_q \leq 208 \text{ and } \hat{y}_s = 198$$

The eigenvalues of D are (10.553, 3.557, 0.979). Recall that the matrix T is given by $T = D_2^{(-1/2)} Q'BOD_2^{(-\frac{1}{2})}$

$$T = \begin{bmatrix} 0.3078 & 0 & 0 \\ 0 & 0.5302 & 0 \\ 0 & 0 & 0.0103 \end{bmatrix} \begin{bmatrix} 0.6428 & 0.6938 & 0.3248 \\ -0.3497 & -0.1115 & 0.9302 \\ 0.6816 & -0.7115 & 0.1710 \end{bmatrix} BOD_2^{-\frac{1}{2}}$$

$$= \begin{bmatrix} -2.0338 & -1.4200 & -0.7715 \\ -1.4100 & -1.4566 & -1.3899 \\ -0.7715 & -1.3899 & -1.4861 \end{bmatrix}$$

with eigenvalues of T being (−4.0617, −0.9945, 0.08017).

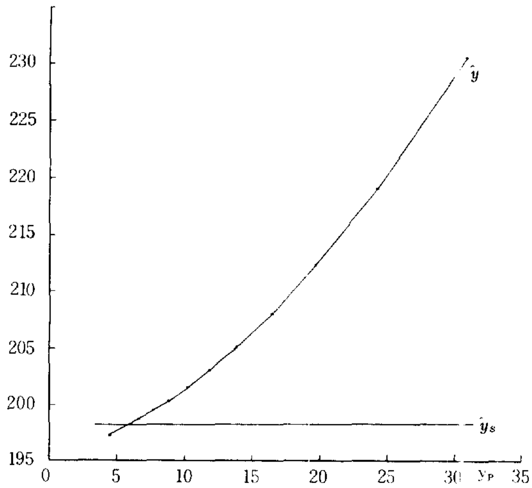

Figure 5.6

Figure 5. 6 gives the values of the estimated responses, $\hat{y}_p$ = 16.3, $\hat{y}_s$ = 198 and $\hat{y}_q$ = 207.9 at the coordinates, x_1 = 6.7, x_2 = −9.9 and x_3 = 0.3.

V. CONCLUSION AND REMARKS

The algorithm associated with the exploration of triple response systems was given in Chapter 2. In that algorithm, it was not clear how to choose a proper Lagrangian multiplier μ by trial-and-error method. But it was possible to choose a proper μ by tracing a locus of $\hat{y}_s$ at the various values of μ, since the relationship between the Lagrangian multiplier γ and $\hat{y}_q$ at any value of μ is known.

In Chapter 3 $\hat{y}_s$ was in linear form and thus μ was uniquely determined by equation (3. 2) and (3.4) after the working region of γ was calculated. Hence, the optimal conditions were easily found by generating simple two dimensional plots.

In case matrices C and D are indefinite, feasibility check method was used. The fact that the number of independent variables in the system is restricted would be the weakness of this approach. Since the shape of the contour system in the estimated feasible region can be shown, it is useful for the users to understand the response system.

The method presented in this paper involved optimization of one response variable subject to constraints on the remaining response variables. Often, however, the goal may be the attainment of the best balance among several different response variables. Derringer [13] presented a disirability function approach to solve this problem. His paper illustrated how several response variable could be transformed into a disirability function, which could be optimized by a univariate technique. The problem involving the simultaneous optimization of several response variables without primary response function deserves further study.

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