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ESTIMATION OF RELIABILITY IN A MULTICOMPONENT STRESS-STRENGTH MODEL IN WEIBULL CASE*

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요 약

동일한 부품 K개를 갖고 있으며, 그 중에서 S개 이상의 스트렝스(strength)가 스트레스(stress) 보다 크게 될 경우 신뢰성이 유지되는 시스템에서 스트레스와 스트렝스가 모두 와이블(weibull)분포를 하고 있을 때의 시스템 신뢰성을 고찰하였다.

2 절에서는 시스템 신뢰성의 최소분산불편추정량(MVU estimator)을 구하였고,

3 절에서는 최소분산불편추정량의 점근분포(asymototic distribution)를 구하고 표본크기가 클때 시스템 신뢰성의 최소분산불편추정량과 최우추정량(MLE)과의 관계를 구하였으며,

4 절에서는 시스템 신뢰성의 일양최적불편신뢰구간(uniformly most accurate unbiased confidence interval)을 구하였고,

5 절에서는 몬테 카를로 씨뮤레이션(Monte Carlo Simulation)을 사용하여 작은 표본에서의 최우추정량과 최소분산불편추정량의 편기(bias)와 평균자승오차(MSE)를 비교하였고

6 절에서는 결과를 간단히 요약하고 본 논문을 더 확장할 경우에 문제점을 제시하였다.

ABSTRACT

A stress-strength model is formulated for s of k systems consisting of identical components. We consider minimum variance unbiased (MVU) estimation of system reliability for data consisting of a random sample from the stress distribution and one from the strength distribution when the two distirubtions are Weibull with unknown scale parameters and same known shape parameter. The asymptotic distribution of MVU estimate of system reliability in the model is obtained by using the standard asymptotic properties of the maximum likelihood estimate of system reliability and establishing their equivalence. Uniformly most accurate unbiased confidence intervals are also obtained for system reliability. Empirical comparison of the two estimates for small samples is studies by Monte Carlo simulation.

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I. INTRODUCTION

Let the random variables X_o, Y_{io}, . . ., Y_{ko} be independent, F(x) be the continuous cumulative distribution function (cdf) of X_o, and G(y) be the common continuous cdf of Y_{io}, i = 1, . . ., k. This paper concerns optimal point and interval estimation procedures for the probability function (1. 1) $R_{s,k}$ = Prob (at least s of the Y_{10}, . . , Y_{ko} exceed X_o)

$$= \sum_{t=s}^{k}\binom{k}{t} \int_{-\infty}^{\infty}[1-G(x)]^t G(x)^{k-t} dF(x)$$

assuming that F and G are Weibull distributions with unknown scale parameters and same known shape parameter. And also assuming that independent random samples X_1, . . ., X_{n1} and Y_1 . . ., Y_{n2} are available from F and G, respectively.

This problem originated in the context of reliability as extension to multicomponent systems of the stress – strength model whose single component versions have been considered by Church and Harris [1], Enis and Geisser [2] and others. Suppose a system, with k identical components, functions if s(1 ≤ s ≤ k) or more of the components simulaneously operate. In its operating environment, the system is subjected to a stress X_o which is a random variable with cdf F. The strengths of the components, that is the minimum stresses to cause failure, are independent and identically distributed random variables with cdf G. Then the system reliability, which is the probability that the system does not fail, is the function $R_{s,k}$ given in (1. 1). The particular cases s = 1 and s = k correspond, respectively, to parallel and series systems. Typically,, the components are mass produced and the data on strengths Y_i, . . ., Y_{n2} can be generated from laboratory load tests on a random sample of the components. Also, the data of stress x_i, . . ., X_{n1} can be obtained from a simulation of conditions of the operating environment. A strong point in favor of component testing is that the data may be used for inference on the reliability of any s out of k system.

Several different systems may be considered at one time and the same data used to draw inferences about the resulting system reliabilities. This feature is very important at the design stage and it contrasts markedly with the situation where a completed system must be constructed and tested for each contemplated choice of s and k.

In the present paper, we derive optimal inference procedures for the reliability function $R_{s,k}$ under the parametric model of Weibull distributions for F and G with the same known shape parameter.

In section 2, we consider minimum variance unbiased (MVU) estimation of the system reliability in a multicomponent stress – strength model.

In section 3, the asymptotic distribution of the MVU estimate of the system reliability in a multicomponent stress – strength model is obtained. And comparison between the maximum likelihood estimate (MLE) and MVU estimate of the system reliability in the model is given.

In section 4, the uniformly most accurate unbiased confidence intervals are also obtained for the system reliability of the model.

In section 5, the comparison of the biases and mean squared errors (MSE) between MVU estimate and MLE for small samples is studied by the Monte Carlo simulation.

In section 6, we provide some remarks and indications of possible future work in this paper.

II. MVU ESTIMATION OF $R_{s,k}$

We first derive a convinient expression for the reliability $R_{s,k}$ introdced in (1. 1) for (F, G) having a relation of the form

$$(1-F)^{\frac{1}{\alpha_1^c}} = (1-G)^{\frac{1}{\alpha_2^c}}$$

where F and G can be taken to be Weibull distributions with known shape parameter c. We then proceed to obtain MVU estimates of $R_{s,k}$ when one or both of a_1 and a_2 are unknown.

Letting $\lambda = \alpha_1^c / \alpha_2^c$ and

$I(u) = [B(s, k-s+1)] \int_0^u x^{s-1} (1-x)^{k-s} dx,$

the expression (1. 1) yields

$$(2 \cdot 1) \qquad R_{s,k} = \int_0^1 \sum_{t=s}^{k} \binom{k}{t} \left(U^{\frac{1}{\lambda}}\right)^t \left(1 - U^{\frac{1}{\lambda}}\right)^{k-t} du$$

$$= \int_0^1 I\left(U^{\frac{1}{\lambda}}\right) du$$

$$= 1 - \frac{B(s+\lambda, k-s+1)}{B(s, k-s+1)}$$

(2. 1) is equivalent to

$$(2 \cdot 2) \qquad R_{s,k} = 1 - \frac{(k-s)!}{B(s, k-s+1)} \left[\prod_{j=s}^{k} (\lambda + j) \right]^{-1}$$

Using a partial expantion for the product of reciprocals in the right hand side of (2. 2), we obtain following an alternative expression which is useful in our subsequent analysis:

$$(2 \cdot 3) \qquad R_{s,k} = 1 - [B(s, k-s+1)]^{-1} \sum_{j=0}^{k-s} (-1)^j \binom{k-s}{j} \cdot (s+j+\lambda)^{-1}$$

Henceforce, we assume

$F(x) = 1 - \exp\{-(\alpha_1 x)^c\},$

$G(x) = 1 - \exp\{-(\alpha_2 x)^c\}, \quad 0 < x < \infty$

and consider first the MVU estimation of reliability for the case when a_1 and a_2 are both unknown. Let $2F_1$ $(\alpha, \beta; \gamma; x)$ denote the hypergeometric function of the second kind which is defined by

$$(2 \cdot 4) \qquad 2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} \cdot (1-t)^{\gamma-\beta-1} (1-tx)^{-\alpha} dt$$

$$\gamma > \beta > 0; x < 1$$

Further, we shall write $< a >$ to denote the integer part of a, and a sum $\sum_{i=m}^{n} a_1$ is to be interpreted as 0 if the lower limit m exceeds the upper limit n.

Let $X_1, \ldots, X_{n1}$ and $Y_1, \ldots, Y_{n2}$ be independent random samples from F(x) and G(y), respectively, and let $\lambda = a_1^c / a_2^c$ In order to derive MVU estimate $\tilde{R}_{s,k}$ of $R_{s,k}$, we first consider MVU estimation of the parametric function $\psi_a(\lambda) = (a + \lambda)^{-1}$ where $a > 0$ is a given constant. A simple unbiased estimate of $\psi_a(\lambda)$ is given by $g(X_1 \; Y_1) = a^{-1} I[aX_1^c > Y_1^c]$ where I is the indicator function of the set which appears in its suffix. Since F and G are Weibull distributions which are exponential families, then $T_1 = {}_{i=1} X_i$ and $T_2 = {}_{i=1} Y_1$ are complete sufficient statistics. Hence, we use the Rao – Blackwell theorem [5] and the Lehmann – Scheffe theorem [6], MVU estimate $\tilde{\psi}_a$ of $\psi_a(\lambda)$ is the conditional expectation $\tilde{\psi}_a = E[g(X_1, Y_1 / T_1, T_2]$. Writing $W_1 = X_1^c / T_1$ and $W_2 = Y_1^c$, we have that (W_1, W_2) is independent of T and has probability density function.

$$(2 \cdot 5) \qquad f(w_1, w_2 | t_1, t_2) = (n_1 - 1)(n_2 - 1)(1 - w_1)^{n_1 - 2} \cdot (1 - w_2)^{n_2 - 2}$$

$$0 < w_i < 1, \quad i = 1, \; 2$$

Let $V = T_1 / T_2$, Then $\tilde{\psi}_a = a^{-1} P(W_2 < aVW_1 | T_1 T_2)$

For the case $aV \leq 1$,

$$(2 \cdot 6) \qquad \tilde{\psi}_a = a^{-1} \int_0^1 \int_0^{aVw_1} f(w_1, y | T_1, T_2) dy \, dw_1$$

$$= a^{-1} [1 - (n_1 - 1) \int_0^1 (1 - aVw_1)^{n_2 - 1} (1 - w_1)^{n_2 - 1} (1 - w_1)^{n_1 - 2} dw_1]$$

$$= a^{-1} [1 - 2F_1(1 - n_2, 1; n_1; aV)]$$

In the same manner, for the case aV > 1,

$$(2 \cdot 7) \qquad \tilde{\psi}_a = a^{-1} \int_0^1 \int_{\frac{w_2}{aV}}^1 f(x, w_2 | T_1, T_2) dx \, dw_2$$

$$= a^{-1} [2F_1(1 - n_1, 1; n_2; \frac{1}{aV})]$$

Expression (2. 6) and (2. 7) together provide the MVU estimate of $\psi_a(\lambda)$. Finally, from (2. 3) we note that $1-R_{s,k}$ is a linear combination of the parametric functions $\psi_a(\lambda)$, a=s, . . ., k. Substitution of the preceding results in this linear function and simplification yield

$$(2\cdot8)\quad \widetilde{R}_{s,k}=1-[B(s,k-s+1)]^{-1}\cdot\Big[\sum_{j=0}^{r_1}(-1)^j\binom{k-s}{j}(s+j)^{-1}\cdot\{1-2F_1[1-n_2,1;n_1;(s+j)V]\}+\sum_{j=r_2}^{k-s}(-1)^j\binom{k-s}{j}(s+j)^{-1}\cdot 2F1[1-n_1,1;n_2;(s+j)^{-1}V^{-1}]\Big],$$

where $r_1=<\min(k-s,\ V^{-1}-s)>$

and $r_2=<\max(0,\ r_1+1)>$

In practice, when computing $\widetilde{R}_{s,k}$ form (2. 8) one can use tables of the hypergeometric function provided the arguments involved lie within the range of the avilable tables. For $n_1>1$, $n_2>1$, the first arguments are negative integers in which case, the $2F_1$ function reduces to a finite sum [3, p.8]. For example

$$(2\cdot9)\quad 2F1(1-n_2,\ 1;n_1;x)=\sum_{j=0}^{n_2-1}\frac{(1-n_2)^{[j]}}{n_1^{[j]}}X^j,$$

where $a^{[0]}=1,\ a^{[j]}=a(a+1)\cdots\cdots(a+j-1).$

Therefore, computation of $\widetilde{R}_{s,k}$ can also be accomplished through summation of finite serieses. But, for large sample sizes, it would be quite laborious to compute. Then we look for a reasonable approximation and the asymptotic distribution of $\widetilde{R}_{s,k}$ will be investigated in section 3.

Now, we consider the MVU estimate of $R_{s,k}$ for the case when a_1 is known and a_2 is unknown. This corresponds to a situation where an extensive simulation experiment or physical theory is available which provides almost complete information about the stress distribution. Let $Y_1,\ldots,Y_n$ be a random sample from

$$G(y)=1-\exp\{-(\alpha_2 y)^c\}$$

So that $T=\sum_{i=1}^{n}Y_i^c$ is a complete sufficient statistic for α_2^c. Taking $g(Y_1)=a^{-1}\exp(-\alpha_1^c Y_1^a/a)$ as the initial unbiased estimate for $\psi_a=(a+\lambda)^{-1}$ and as same method in the case when a_1 and a_2 are both unknown, the MVU estimate $R^*_{s,k}$ of $R_{s,k}$ is obtained as

$$(2\cdot10)\quad R^*_{s,k}=1-[B(s,k-s+1)]^{-1}\{\sum_{j=0}^{k-s}(-1)^j\binom{k}{j}\cdot(s+j)^{-1}M[1;n;-T\alpha_1^c(s+j)^{-1}]\}$$

where the confluent hypergeometric function of the first kind is defined by

$$M(\alpha;\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}e^{tx}dt.$$

III. ASYMPTOTIC DISTRIBUTION

Here, we investigate the asymptotic distribution of $\widetilde{R}_{s,k}$ for the case when both a_1 and a_2 are unknown. The results for the case of known a_1 are completely parallel and are therefore omitted. To derive the limiting distribution of $\widetilde{R}_{s,k}$ directly from (2. 8) it is tedious because of its complexity. To circumvent this difficulty, consider the standard asymptotic property of maximum likelihood estimate (MLE) of $R_{s,k}$ and then establish its asymptotic equivalance with $\widetilde{R}_{s,k}$. In this process, we also obtain a first order correction term for removal of bias of the MLE.

The MLE of (α_1^c, α_2^c) is given by

$$\left(\hat{\alpha}_1^c=\frac{n_1}{\sum_{i=1}^{n_1}X_i^c}\quad \hat{\alpha}_2^c=\frac{n_2}{\sum_{i=1}^{n_2}Y_i^c}\right)$$

Now, we write $\widetilde{R}_{s,k}^{(n)}$ for the MVU estimate given in (2. 8) and $\hat{R}_{s,k}^{(n)}$ for the MLE of $R_{s,k}$ where $n=n_1+n_2$ is the combined sample size. By the invariance property of MLE and (2. 2), the MLE $\hat{R}_{s,k}^{(n)}$ of $R_{s,k}$ is obtained as

(3·1) $\hat{R}^{(n)}_{s,k}=1-\{(k-s)!\,[B(s,k-s+1)]^{-1}\,[\prod_{j=s}^{k}(j+\hat{\lambda}_n)]^{-1}\}$

where $\hat{\lambda}_n=n_1\sum_{i=1}^{n_2}Y_i^c\,/\,(n_2\sum_{i=1}^{n_1}X_i^c)$

The limiting distribution of the MLE is obtained by the following procedure where $\mathcal{L}$ denotes convergence in distribution. $\sum_{i=1}^{n_1} x_i^c/n_1$ is distributed as gamma distribution with parameters n_i and $n_1\alpha_1^c$. Then by the central limit theorem

$$\frac{\dfrac{\sum_{i=1}^{n_1}X_i^c}{n_1}-\dfrac{1}{\alpha_1^c}}{\sqrt{\dfrac{1}{n_1\alpha_1^{2c}}}}\xrightarrow{\mathcal{L}}N(0,1)$$

So,

(3·2) $n^{\frac{1}{2}}(\sum_{i=1}^{n_1}X_i^c/n_1-1/\alpha_1^c)\xrightarrow{\mathcal{L}}N(0,1/[\alpha_1^{2c}\cdot\gamma])$

where $n_1/n\to\gamma$ as $n\to\infty$, $0<\gamma<1$.

As the same method

(3·3) $n^{\frac{1}{2}}(\sum_{i=1}^{n_2}Y_i^c/n_2-1/\alpha_2^c)\xrightarrow{\mathcal{L}}N(0,1/[\alpha_2^{2c}(1-\gamma)])$

and, (3. 2) and (3. 3) are independent. Since $\lambda=\alpha_1^c/\alpha_2^c$ is a function of α_1^c, α_2^c with continuous partial derivatives, the result 6a.2.6 of Rao [6, p.387] gives

(3·4) $n^{\frac{1}{2}}(\hat{\lambda}_n-\lambda)\xrightarrow{\mathcal{L}}N(0,\lambda^2/[\lambda(1-\gamma)])$

Note that $1-R_{s,k}=r\exp(g(\lambda))$ with $r=(k\text{-}s)!\cdot[B(s,k-s+1)]^{-1}$, $g(\lambda)=\sum_{j=s}^{k}\log(j+\lambda)^{-1}$, and using the above argument once more, we have

(3·5) $n^{\frac{1}{2}}(\hat{R}^{(n)}_{s,k}-R_{s,k})\xrightarrow{\mathcal{L}}N(0,\alpha^2_{s,k})$

where $\alpha^2_{s,k}=[\gamma(1-\gamma)]^{-1}(1-R_{s,k})^2\lambda^2[\sum_{j=s}^{k}(j+\lambda)^{-1}]^2$

From (3. 5) we can derive the limiting distribution of MLE $\hat{R}^{(n)}_{s,k}$.

To derive an asymptotic expansion of the MVU estimate $\tilde{R}^{(n)}_{s,k}$, having $\hat{R}^{(n)}_{s,k}$ as the leading term, we first note that, due to (2. 3)

(3·6) $\hat{R}^{(n)}_{s,k}=1-\sum_{j=0}^{k-s}l_j\hat{\psi}^{(n)}_{s+j}$, $\tilde{R}^{(n)}_{s,k}=1-\sum_{j=0}^{k-s}l_j\tilde{\Psi}^{(n)}_{s+j}$

where $l_j=[B(s,k-s+1)]^{-1}(-1)^j\binom{k-s}{j}$, and $\hat{\psi}^{(n)}_a$ and $\tilde{\Psi}^{(n)}_a$ are the MLE and the MVU estimate of $\Psi_a=(a+\lambda)^{-1}$, respectively. Since the coefficients ℓ_j are fixed constant irrespective of the sample size, it suffices to show the relation between $\hat{\Psi}_a$ and $\tilde{\psi}^{(n)}_a$ as $n=(n_1+n_2)\to\infty$, $n_1/n\to\gamma$, $0<\gamma<1$ and for a fixed a.

Theroem 3·1: Let $Z_{a\,n}=an_1/(n_2\hat{\lambda}_n)$ where $\hat{\lambda}_n=n_1\sum_{i=1}^{n_2}Y_i^c/(n_2\sum_{i=1}X_i^c)$. Then as $n=(n_1+n_2)\to\infty$ such that $n_1/n=\gamma+0(n^{-1})$, $0<\gamma<1$, we have with probability 1

(3·7) $\tilde{\psi}^{(n)}_a=\hat{\psi}^{(n)}_a+n^{-1}U_{a,n}+0(n^{-1})$

where

(3·8) $U_{a,n}=a^{-1}[Z^2_{a,n}(n_2/n)^2-Z_{a,n}(n_1/n)^2]\cdot[Z_{a,n}(n_2/n)+(n_1/n)]^{-3}$

Proof: We consider first the two leading terms in an asymptotic expansion of the integral

$$\phi_{n_1,n_2}(z)=(n_1-1)\int_0^1(1-zw)^{n_2-1}(1-w)^{n_1-2}\,dw$$

for $0<z<1$ and $n=(n_1+n_2)\to\infty$ such that $n_1/n\to\gamma$, $0<\gamma<1$. Expanding

$$h_n(w,z)=[(n_2-1)/n]\log(1-zw)+[(n_1-2)/n]\log(1-w)$$

about w=0 and considering $\exp[n\,h_n(w,z)]$, we have

$$\phi_{n_1,n_2}(z)=\frac{n_1}{n}\frac{1}{\beta_n}-\frac{1}{n}\left[\frac{1}{\beta_n}-\frac{n_1}{n}\frac{(z+2)}{\beta_n^2}+\frac{n_1}{n}\frac{[Z^2(n_2/n)+(n_1/n)]}{\beta_n^3}\right]+0\left(\frac{1}{n}\right)$$

where $\beta_n=(n_2/n)z+(n_1/n)$.

The same analysis also provides an expansion of $\phi_{n2,\,n1}(1/z)$ and these expansions are uniform in a neighborhood of points Z_0 in $0<Z_0\le 1$ and $1\le Z_0<\infty$, respectively.

From the expression (2. 6) and (2. 7), we note that the MVU estimate $\tilde{\psi}^{(n)}_a$ of ψ_a is given by

$$(3\cdot9)\qquad \tilde{\Psi}_a^{(n)}=a^{-1}[1-\phi_{n_1,n_2}(aVn)] \text{ for } aVn\leq 1$$
$$=a^{-1}[\phi_{n_2,n_1}(1/aVn)] \text{ for } aVn>1$$

By the strong law of large numbers,

$$aVn=a\frac{\sum_{i=1}^{n_1} X_i^c/n_1}{\sum_{i=1}^{n_2} Y_i^c/n_2}\cdot\frac{n_1}{n_2}\rightarrow (a/\lambda)[\gamma/(1-\gamma)]$$

with probability 1 as $n\rightarrow\infty$. Recognizing that

$$a^{-1}\left[1-\frac{n_1}{n}\left(\frac{n_2}{n}aVn+\frac{n_1}{n}\right)^{-1}\right]=(a+\hat{\lambda}_n)^{-1},$$

where $\hat{\lambda}_n$ is the MLE of λ and simplifying this expression, this theorem is proved.

Now, we consider the relation between $\tilde{R}_{s,k}^{(n)}$ and $\hat{R}_{s,k}^{(n)}$. Let $U_n = \sum_{j=0}^{k-s}\ell_j \quad U_{s+j,n}$. We have from (3·6)and (3·7)

$$(3\cdot10)\qquad \tilde{R}_{s,k}^{(n)}=\hat{R}_{s,k}^{(n)}-(1/n)U_n+0(n^{-1})$$

By the strong law of large numbers, $Z_{a,n}\xrightarrow{a.s.} a\gamma\,[\lambda(1-\gamma)]^{-1}$ a.s $n\rightarrow\infty$ which implies that U_n converges to a finite constant and, hence, that $n^{\frac{1}{2}}(\tilde{R}_{s,k}^{(n)}-\hat{R}_{s,k}^{(n)})\xrightarrow{a.s.}0$ thus the limiting distribution stated in (3·5) for $\hat{R}_{s,k}^{(n)}$ also holds for $\tilde{R}_{s,k}^{(n)}$ since $\tilde{R}_{s,k}^{(n)}$ is unbiased for $R_{s,k}$ the term $\frac{1}{n}U_n$ in (3·10) provides an estimate of the first -- order correction term for bias in the MLE.

IV. CONFIDENCE INTERVAL

To derive a confidence interval for the reliability $R_{s,k}$ given in (2·1), we first note that $2\alpha_1^c T_1$ and $2\alpha_2^c T_2$ are independent X_2 with $2n_1$ and, $2n_2$ degrees of freedom, respectively, where $T_1=\sum_{i=1}^{n_1}X_i^c$ and $T_2=\sum_{i=1}^{n_2}Y_i^c$ Using the method in comparing the ratio of variances of two normal populations in Lehmann [4, P. 170], the confidence interval for $\lambda=\alpha_1^c/\alpha_2^c$ can be obtained. Let

$$(4\cdot1)\qquad W_1=2\alpha_1^c T_1,\; W_2=2\alpha_2^c T_2\;,$$

and

$$(4\cdot2)\qquad W=W_2/(W_1+W_2)$$

Then the uniformly most accurate unbiased acceptance region for W with confidence coefficient $1-\alpha$ is given by

$$(4\cdot3)\qquad c_1<W<c_2$$

Let $W=Y/(1+Y)$ where $Y=W_2/W_1$. Since $(2n_1)Y/(2n_2)$ has F distribution with degrees of freedom $2n_2$ and $2n_1$, the distribution of W is the beta distribution as following:

$$B_{n_2,n_1}(w)=\frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\,\Gamma(n_2)}w^{n_2-1}(1-w)^{n_1-1}$$

where $0<w<1$. Using the conditions (5) and (6) in Lehmann [4, P 161], the relations

$$E(W)=\frac{n_2}{n_1+n_2}$$

and

$$wB_{n_2,n_1}(W)=\frac{n_2}{n_1+n_2}B_{n_2+1,\,n_1}(w),$$

we can obtain the following equation

$$(4\cdot4)\qquad \int_{c_1}^{c_2}B_{n_2,n_1}(w)\,dw=\int_{c_1}^{c_2}B_{n_2+1,n_1}(w)\,dw=1-\alpha\,,$$

From (4. 4) we can determine c_1 and c_2.

By the relations (4·1), (4·2) and (4·3). the uniformly most accurate unbiased (UMAU) confidence interval for λ with confidence coefficient $1-\alpha$ is given by

$$(4\cdot5)\qquad \underline{\lambda}=\frac{(1-c_2)}{c_2}\frac{T_2}{T_1}<\lambda<\frac{(1-c_1)}{c_1}\frac{T_2}{T_1}=\bar{\lambda}$$

with (4. 4).

The equation (2. 2) shows that $R_{s,k}$ is a monotone, strictly increasing, function of the parammeter λ which we denote explicitly by $R_{s,k}(\lambda)$. From monotonicity in λ and (4. 5) the UMAU confidence interval for $R_{s,k}(\lambda)$ is given by

$$(4\cdot6)\qquad R_{s,k}(\underline{\lambda})<R_{s,k}(\lambda)<R_{s,k}(\bar{\lambda})$$

A one-sided UMAU confidence interval for $R_{s,k}$ is obtained in the similar way from the one-sided interval for λ.

Here we note that neither the MVU estimate nor the MLE of system reliability is useful in setting the UMAU confidence interval.

We also remark that the components are tested in unassembled form rather than as a completed system, the confidence statement (4. 6) holds simultaneously for all s and k. These simultaneous confidence intervals based on information on components are useful for selecting the form of the system.

V. EMPRICAL COMPARISON IN SMALL SAMPLES

The large sample theory presented in section 3 shows that the MVU estimate and the MLE of $R_{s,k}$ given in (2. 8) and (3. 1) are asymptotically equivalant.

Now, in this section we investigate their relative performance in small sample $n_1 = 5$, $n_2 = 5$ through the Monte Carlo simulation. This corresponds to a situation where the sampling cost is very expensive.

Estimates of the mean squared error (MSE) and bias were obtained from 5000 trials for the two out of four and one out of three systems with λ = 2, 3, 4, 5, 6. In each situation a trial consisted of generating 10 uniform (0, 1) random numbers, i.e. RN_i, i=1, . . ., 10, and their transforming to $X_i^c = -\log(RN_i)$, $i \leq 5$, and $Y_i^c = -\lambda \log(RN_i)$, i = 6, 7, . . ., 10. From these, the value of the MVU estimate $\tilde{R}_{s,k}$ was obtained using (2. 8) and (2.9) and the value of the MLE $\hat{R}_{s,k}$ from (3. 1). The true value of $R_{s,k}$ was computed from (2. 2) for each λ, s and k. The results on the estimated MSE and bias appear in the following Table 5. 1.

Table 5. 1. Estimates of Bias and MSE: $n_1 = n_2 = 5$

Reliability			Bias		Mean squared error	
(s,k)	λ	$R_{s,k}$	MVU	MLE	MVU	MLE
(1, 3)	2	.90000	-.00130	-034002	.01442	.01446
	3	.95000	.00005	.02783	.00636	.00784
	4	.97143	.00020	.02204	.00289	.00434
	5	.98214	.00162	-.01613	.00124	.00226
	6	.98810	-.00039	-.01460	.00095	.00184
(2, 4)	2	.80000	-.00279	-.03146	.03027	.02491
	3	.88571	-.00391	-.03637	.01838	.01750
	4	.92857	.00242	-.02950	.00937	.01046
	5	.95238	-.00094	-.02912	.00626	.00791
	6	.00040	-.00040	-.02482	.00414	.00569

Although MVU estimate $\tilde{R}_{s,k}$ is known to be unbiased, its estimated bias is recorded as a check on the computations. In Table 5.1, the MSE of both estimates appear to be nearly equal, but the bias in MLE is considerably larger than the bias in MVU estimate. Hence we conclude that in small sample size, the MVU estimate of $R_{s,k}$ is better than the MLE of $R_{s,k}$.

VI. CONCLUDING REMARKS

In the preceding section we showed that the MVU estimate $\tilde{R}_{s,k}$ of the reliability $R_{s,k}$ is better than MLE $\hat{R}_{s,k}$ in small sample sizes But, as the sample size n becomes large these are equivalant. And using the first – order correction term U_n in equation (3.10), we can reduce the error term between $\tilde{R}_{s,k}$ and $\hat{R}_{s,k}$.

We conclude by remarking that the strong point of component testing that the several different systems may be considered at one time and the same data used to draw inferences about the resulting system reliabilities. And in this procedure, the confidence statement in section 4 holds simulataneously for all s and k, and this is useful for selecting the form of the system.

In this paper, we investigate the case when the stress and the strength have Weibull distributions. But it can be considered when the two have not the same distribution, for example the stress has a Weibull distribution and the strength has an exponential distribution and vice versa. This subject destribution and vice versa. This subject deserves further considerations.

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