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AUTOMORPHISMS AND PROPER BASES IN SPACE OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES REPRESENTED BY DIRICHLET SERIES(*)

By J.K. Srivastava and Rajiv K. Srivastava

1. In this note for simplicity we are considering functions of only two variables, results can easily be extended to any finite number of variables. Let $\mathbb{C}^2$ be the cartesian product of $\mathbb{C}$ with itself equipped with the usual product topology. Let

$$D=\{(\sigma_1+it_1,\ \sigma_2+it_2):\sigma_1<C_1,\ \sigma_2<C_2,\ \text{and}\ -\infty<t_1,\ t_2<\infty\},$$

where C_1 and C_2 are arbitrarily chosen positive real numbers but fixed throughout this note. Let

$$0<\lambda_1<\lambda_2<\cdots<\lambda_m<\cdots\longrightarrow\infty;$$
$$0<\mu_1<\mu_2<\cdots<\mu_n<\cdots\longrightarrow\infty;$$

with

$$(1.1)\qquad \lim_{m\to\infty}\frac{\log m}{\lambda_m}=0=\lim_{n\to\infty}\frac{\log n}{\mu_n}.$$

Consider E to be the class of functions $f:D\to\mathbb{C}$ represented by Dirichlet series

$$(1.2)\qquad f(s_1,\ s_2)=\sum_{m,n=1}^{\infty}a_{mn}e^{\lambda_m s_1+\mu_n s_2},\ (s_1,\ s_2)\in D,$$

and the series (1.2) be convergent in D (on account of (1.1) the series (1.2) will then be absolutely convergent). Equivalently, E is the class of all functions f for which the Dirichlet series (1.2) satisfy

$$(1.3)\qquad \limsup_{m+n\to\infty}\frac{\log|a_{mn}|+\lambda_m C_1+\mu_n C_2}{\lambda_m+\mu_n}\leq 0.$$

In [4], we have studied some of the properties pertaining to the topological structures of the linear space E by endowing it with different topologies, particularly with respect to the locally convex topology ξ on E generated by the family of seminorms $\{\|\cdot\,;\sigma_1;\sigma_2\|,\sigma_1<C_1,\ \sigma_2<C_2\}$ where

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$$\|f;\sigma_1,\sigma_2\|=\sum_{m,n=1}^{\infty} a_{mn} e^{\lambda_m\sigma_1+\mu_n\sigma_2}. \tag{1.4}$$

Let $\sigma_i^{(k)} \longrightarrow C_i$, $k \longrightarrow \infty$, $i=1,2$ and considering the invariant metric

$$\rho(f,g)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{\|f-g;\sigma_1^{(k)},\sigma_2^{(k)}\|}{1+\|f-g;\sigma_1^{(k)},\sigma_2^{(k)}\|}, \tag{1.5}$$

we have shown (see [4]) that (E,ξ) is a Fréchet space.

Let $\delta_{mn}(s_1,s_2)=e^{\lambda_m s_1+\mu_n s_2}$, $m,n\geq 1$. We recall that a double sequence $\{\alpha_{mn}\}$ is said to be *a basis of the subspace* E_0 of E if it is linearly independent and spans E_0. $\{\delta_{mn}\}$ is a basis of E. A double sequence $\{\alpha_{mn}\}\subset E$ which is also a basis for a subspace E_0 of E is said to be *proper basis for* E_0 if for all double sequences $\{a_{mn}\}$ of complex numbers $\sum_{m,n=1}^{\infty} a_{mn}\delta_{mn}$ converges if and only if $\sum_{m,n=1}^{\infty} a_{mn}\alpha_{mn}$ converges.

Following characterization of proper bases has been obtained in [4] :

THEOREM 1.1. *A basis* $\{\alpha_{mn}\}$ *in a closed subspace* E_0 *of* E *is proper if and only if*

$$\limsup_{m+n\to\infty}\frac{\log\|\alpha_{mn};\sigma_1,\sigma_2\|-\lambda_m C_1-\mu_n C_2}{\lambda_m+\mu_n}<0, \text{ and} \tag{1.6}$$

$$\lim_{\substack{\sigma_1\to C_1\\ \sigma_2\to C_2}}\left\{\liminf_{m+n\to\infty}\frac{\log\|\alpha_{mn};\sigma_1,\sigma_2\|-\lambda_m C_1-\mu_n C_2}{\lambda_m+\mu_n}\right\}\geq 0. \tag{1.7}$$

Kamthan and Gautam [2] initiated the study of the spaces of analytic Dirichlet functions. We intend to generalize their results to the case of analytic functions of several complex variables represented by Dirichlet series.

In section 2, we study certain linear homeomorphisms in (E,ξ) and characterize linear homeomorphisms in terms of proper bases. In section 3, we study continuous linear operators from (E,ξ) into itself and form (E,ξ) onto the closed subspace $(F,\mathscr{F})$ of (E,ξ). ($\mathscr{F}$ is stronger than ξ). The last section 4 is devoted to the study of simultaneous automorphisms in terms of proper bases.

2. Linear homeomorphisms

The main result of this section is the characterization of proper bases in terms of linear homeomorphisms. We first prove the following lemma:

LEMMA 2.1. *Let T be a linear mapping of E into itself. Then T is continuous on (E, ξ) if and only if for each $\sigma_1<C_1$ and $\sigma_2<C_2$ there correspond $\delta_1<C_1$ and $\delta_2<C_2$ such that T is a continuous mapping from $E(\delta_1, \delta_2)$ into $E(\sigma_1, \sigma_2)$.*

PROOF. Let $\sigma_1<C_1$, $\sigma_2<C_2$ be given. Then from the hypothesis we can find $\delta_1<C_1$, $\delta_2<C_2$ such that T maps continuously $E(\delta_1, \delta_2)$ into $E(\sigma_1, \sigma_2)$. Let $f_n \to 0$ in ξ. Hence in particular $\|f_n ; \delta_1, \delta_2\| \to 0$, which implies that $\|Tf_n ; \sigma_1, \sigma_2\| \to 0$. Since $\sigma_1<C_1$ and $\sigma_2<C_2$ are arbitrary, we conclude that T is continuous.

Conversely, assume T be continuous on (E, ξ) but for $\sigma_1<C_1$ and $\sigma_2<C_2$ there do not exist $\delta_1<C_1$ and $\delta_2<C_2$ such that T maps $E(\delta_1, \delta_2)$ continuously into $E(\sigma_1, \sigma_2)$. Thus for every p we can find f_p such that

$$\|f_p ; \sigma_1^{(p)}, \sigma_2^{(p)}\| \leq \frac{1}{p}$$

and

$$\|Tf_p ; \sigma_1, \sigma_2\| \geq 1.$$

Then

$$\begin{aligned}\rho(f_p, 0) &= \sum_{k=1}^{p} \frac{1}{2^k} \frac{\|f_p ; \sigma_1^{(k)}, \sigma_2^{(k)}\|}{1+\|f_p ; \sigma_1^{(k)}, \sigma_2^{(k)}\|} \\ &\quad + \sum_{k=p+1}^{\infty} \frac{1}{2^k} \frac{\|f_p ; \sigma_1^{(k)}, \sigma_2^{(k)}\|}{1+\|f_p ; \sigma_1^{(k)}, \sigma_2^{(k)}\|} \\ &< \frac{2}{p+1},\end{aligned}$$

i.e., $\rho(f_p, 0) \to 0$ but $T f_p \not\to 0$ in ξ, a contradiction.

THEOREM 2.1. *Let $\{\alpha_{mn}\}$ be a double sequence in (E, ξ). Then there exists a continuous linear mapping $T : E \to E$ such that $T\delta_{mn} = \alpha_{mn}$ if and only if* (1.6) *holds.*

PROOF. Let T be a continuous linear mapping with $T\delta_{mn} = \alpha_{mn}$ and $\sigma_1<C_1$, $\sigma_2<C_2$ be given. Then by Lemma 2.1 we can find $\delta_1<C_1$, $\delta_2<C_2$ such that T maps $E(\delta_1, \delta_2)$ continuously into $E(\sigma_1, \sigma_2)$, i.e., there exists K such that

$$\begin{aligned}\|\alpha_{mn} : \sigma_1, \sigma_2\| &= \|T\delta_{mn} : \sigma_1, \sigma_2\| \\ &\leq K\|\delta_{mn} ; \delta_1, \delta_2\| = Ke^{\lambda_m \delta_1 + \mu_n \delta_2}.\end{aligned}$$

Thus

$$\limsup_{m+n\to\infty} \frac{\log\|\alpha_{mn} ; \sigma_1, \sigma_2\| - \lambda_m C_1 - \mu_n C_2}{\lambda_m + \mu_n}$$

$$\leq \limsup_{m+n\to\infty} \frac{\log K+\lambda_m(\delta_1-C_1)+\mu_n(\delta_2-C_2)}{\lambda_m+\mu_n}$$

$$\leq max(\delta_1-C_1,\ \delta_2-C_2)<0.$$

Conversely, suppose (1.6) holds and $\sigma_1<C_1$ and $\sigma_2<C_2$. We can then find $\varepsilon>0$ such that

$$\limsup_{m+n\to\infty} \frac{\log\|\alpha_{mn}\,;\,\sigma_1,\ \sigma_2\|-\lambda_m C_1-\mu_n C_2}{\lambda_m+\mu_n}<-\varepsilon,$$

i.e. , we can find K_1 such that

$$\|\alpha_{mn}\,;\,\sigma_1,\ \sigma_2\|\leq K_1 e^{\lambda_m(C_1-\varepsilon)+\mu_n(C_2-\varepsilon)},\ m,\ n\geq 1. \tag{2.1}$$

Let $f=\sum_{m,n=1}^{\infty} a_{mn}\,\delta_{mn}\in E$. For $\eta<\varepsilon$, owing to (1.3) we can find K_2 such that

$$|a_{mn}|\leq K_2\ e^{\lambda_m(-C_1+\eta)+\mu_n(-C_2+\eta)},\ m,\ n\geq 1. \tag{2.2}$$

From (2.1) and (2.2) we see that $\|\sum_{m,n=1}^{\infty} a_{mn}\ \alpha_{mn}\,;\,\sigma_1,\ \sigma_2\|$ is convergent, and hence $\sum_{m,n=1}^{\infty} a_{mn}\,\alpha_{mn}$ converges in ξ. We define

$$T(f)=T(\sum_{m,n=1}^{\infty} a_{mn}\,\delta_{mn})=\sum_{m,n=1}^{\infty} a_{mn}\,\alpha_{mn}.$$

Clearly T is well defined and is linear. Also, in view of (2.1)

$$\|Tf\,;\,\sigma_1,\ \sigma_2\|\leq K_1 \sum_{m,n=1}^{\infty} |a_{mn}|\,e^{\lambda_m(C_1-\varepsilon)+\mu_n(C_2-\varepsilon)}$$

$$=K_1\|f\,;\,C_1-\varepsilon, C_2-\varepsilon\|,$$

i.e., T maps $E(C_1-\varepsilon,\ C_2-\varepsilon)$ continuously into $E(\sigma_1,\ \sigma_2)$. Hence by Lemma 2.1 T maps continuously E into itself.

THEOREM 2.2. *If T is a linear homeomorphism of $(E,\ \xi)$ into itself, then $\{T\delta_{mn}\}$ is a proper basis in some closed subspace E_0 of E. Conversely, if the double sequence $\{\alpha_{mn}\}$ is a proper basis in some closed subspace E_0 of E, then there exists a linear homeomorphism T of E into E_0 such that $T\ \delta_{mn}=\alpha_{mn}$, $m,\ n\geq 1$.*

PROOF. Let T be a linear homeomorphism of $(E,\ \xi)$ into itself and E_0 be the range of T. Clearly E_0 is a closed subspace. Let $T\delta_{mn}=\alpha_{mn}$. Suppose $f\in E_0$ then $T^{-1}f\in E$ and so $T^{-1}f=\sum_{m,n=1}^{\infty} a_{mn}\,\delta_{mn}$ for some sequence $\{a_{mn}\}\subset\mathbb{C}$. By continuity of T it follows that $f=\sum_{m,n=1}^{\infty} a_{mn}\alpha_{mn}$, i.e., $\{\alpha_{mn}\}$ spans E_0. Since both T and T^{-1} are continuous, it easily follows that $\{\alpha_{mn}\}$ is a proper basis in E_0.

Conversely, let $\{\alpha_{mn}\}$ be a proper basis for some closed subspace E_0 of E. Then by Theorem 1.1 and Theorem 2.1 there exists a continuous linear mapping of E into itself such that $T\delta_{mn}=\alpha_{mn}$. From continuity and linearity of T we can easily see that T is one-one mapping of E onto E_0. Now E_0, being a closed subspace of E, is complete hence applying Open Mapping Theorem [1, Theorem 2, p.57] we conclude that T is a homeomophism.

As an easy consequence of the above result, we get

COROLLARY 2.1. *Let E_1 and E_2 be two closed subspaces of E. If $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ are, respectively, proper basis for E_1 and E_2, then there exists a linear homeomorphism T from E_1 onto E_2 such that $T\alpha_{mn}=\beta_{mn}$. Conversely if T is linear homeomorphic mapping of E_1 onto E_2 and $\{\alpha_{mn}\}$ is a proper basis for E_1, then $\{T\alpha_{mn}\}$ is a proper basis for E_2.*

3. Certain continuous linear operators

Now we deal with two topological linear spaces, one is the space (E, ξ) and the other is the space F as a subspace of E equipped with the topology $\mathscr{F}$, which is stronger than the relative topology induced on F by ξ, and with respect to $\mathscr{F}$, F is complete metrizable locally convex space. Naturally this topology $\mathscr{F}$ can completely be determined by the sequence of seminorms. Let $\{\|f\|_p,\ p\geq 1\}$ be the required family of seminorms. Since $\mathscr{F}$ is stronger, implies that for every $\sigma_1<C_1$, $\sigma_2<C_2$ we can find a constant K (depending on σ_1, σ_2) and $p\geq 1$ such that

$$\|f;\sigma_1,\ \sigma_2\|\leq K\|f\|_p,\ f\in F. \tag{3.1}$$

In this section we construct certain continuous linear operators from E into itself and from E into F.

THEOREM 3.1. *Let $\{\alpha_{mn}\}$ be a proper basis in E and $\{\phi_{mn}\}$ be a double sequence which satisfies*

$$\limsup_{m+n\to\infty}\frac{\log\|\phi_{mn};\sigma_1,\ \sigma_2\|-\lambda_m C_1-\mu_n C_2}{\lambda_m+\mu_n}<0, \tag{3.2}$$

for every $\sigma_1<C_1,\ \sigma_2<C_2$ *and if*

$$f=\sum_{m,n=1}^{\infty} a_{mn}\,\alpha_{mn}\in E, \quad \textit{then}$$

$$(3.3)\qquad Pf=\sum_{m,n=1}^{\infty} a_{mn}\,\phi_{mn}$$

defines a continuous linear operator on E.

PROOF. Let $f\in E$, then there exists a double sequence $\{a_{mn}\}$ such that $f=\sum_{m,n=1}^{\infty} a_{mn}\,\alpha_{mn}$, moreover since $\{\alpha_{mn}\}$ is proper basis, $\{a_{mn}\}$ will satisfy (1.3). Now if we follow the proof of sufficiency part of Theorem 2.1, for every $\sigma_1<C_1$, $\sigma_2<C_2$ we can easily find $\delta_1<C_1$, $\delta_2<C_2$ and K_1 such that

$$(3.4)\qquad \|\phi_{mn}\,;\sigma_1,\ \sigma_2\|\le K_1 e^{\lambda_m\delta_1+\mu_n\delta_2},\quad m,\ n\ge 1$$

and $\sum_{m,n=1}^{\infty} a_{mn}\,\phi_{mn}$ is (absolutely) convergent in (E,ξ). Thus the mapping P is well defined by (3.3) and is linear.

Let $f=\sum_{m,n=1}^{\infty} a_{mn}\alpha_{mn}$, since $\{\alpha_{mn}\}$ is a proper basis, therefore, $g=\sum_{m,n=1}^{\infty} a_{mn}\,\delta_{mn}$ exists in E. Moreover there exists a linear homeomorphism T (see Theorem 2.1) such that $T^{-1}\alpha_{mn}=\delta_{mn}$, *i.e.*, $T^{-1}f=g$. Hence for δ_1, δ_2 we can find δ_1', δ_2' and K_2 such that

$$(3.5)\qquad \|g\,;\delta_1,\ \delta_2\|=\|T^{-1}f\,;\delta_1,\ \delta_2\|\le K_2\|f\,;\delta_1',\ \delta_2'\|.$$

Using (3.4) and (3.5) we get

$$\|Pf\,;\sigma_1,\ \sigma_2\|\le K_1\|g\,;\delta_1,\ \delta_2\|\le K_1K_2\|f\,;\delta_1',\ \delta_2'\|,$$

where K_1 and K_2 are independent of f, hence P is continuous.

THEOREM 3.2. *Let* $\{\alpha_{mn}\}$, $\{\phi_{mn}\}$ *and* P *be as defined in Theorem* 3.1. *Then* P *is continuous operator of* E *into* F *if and only if* $\{\phi_{mn}\}\subset F$ *and*

$$(3.6)\qquad \limsup_{m+n\to\infty}\frac{\log\|\phi_{mn}\|_p-\lambda_m C_1-\mu_n C_2}{\lambda_m+\mu_n}<0,\quad (p\ge 1).$$

Moreover the expansion (3.3) *converges in* $(F,\mathscr{F})$ *for every* $f\in E$.

PROOF. Fix an integer p, then from (3.6) it follows that there exist $\eta>0$ and K_1 such that

$$(3.7)\qquad \|\phi_{mn}\|_p\le K_1 e^{\lambda_m(C_1-\eta)+\mu_n(C_2-\eta)},\quad m,\ n\ge 1.$$

Let $f=\sum_{m,n=1}^{\infty} a_{mn}\alpha_{mn}$, then $\{a_{mn}\}$ will satisfy (1.3). Choose $\varepsilon<\eta$ then there exists K_2 such that

$$|a_{mn}|\leq K_2 e^{\lambda_m(\varepsilon-C_1)+\mu_n(\varepsilon-C_2)},\ m,\ n\geq 1. \tag{3.8}$$

From (3.7) and (3.8) we get that $\sum_{m,n=1}^{\infty} a_{mn}\phi_{mn}$ converges in $\mathscr{F}$. Hence the mapping P from E into F is well defined and is linear.

Let $g=\sum_{m,n=1}^{\infty} a_{mn}\delta_{mn}$. Since $\{\alpha_{mn}\}$ is proper basis, in view of Theorem 2.1 there exists a linear homeomorphism T such that $T\delta_{mn}=\alpha_{mn}$ (or $T^{-1}\alpha_{mn}=\delta_{mn}$). Therefore for $C_1-\eta$, $C_2-\eta$, we can find δ_1, δ_2 and K_3 such that

$$\begin{aligned}\|g\,;C_1-\eta,\ C_2-\eta\|&=\|T^{-1}f\,;C_1-\eta,\ C_2-\eta\|\\&\leq K_3\|f\,;\delta_1,\ \delta_2\|.\end{aligned} \tag{3.9}$$

Using (3.7) and (3.9), we get

$$\begin{aligned}\|Pf\|_p&\leq K_1\sum_{m,n=1}^{\infty}|a_{mn}|\,e^{\lambda_m(C_1-\eta)+\mu_n(C_2-\eta)}\\&\leq K_1K_3\|f\,;\delta_1,\ \delta_2\|,\end{aligned}$$

i.e., P is continuous.

On the other hand, let P be a continuous, then obviously $\{\phi_{mn}\}\subset F$. We note that if for $\{a_{mn}\}$, $\sum_{m,n=1}^{\infty} a_{mn}\alpha_{mn}$ converges, then $a_{mn}\alpha_{mn}\to 0$ in $(E,\ \xi)$, hence by continuity $a_{mn}\phi_{mn}$ should necessarily converge to 0 in $(F,\ \mathscr{F})$. Further suppose (3.6) is not true for some p. We can find, for a sequence $\{r_k\}$ of real numbers increasing to 0, subsequence of integers $\{m_k\}$ and $\{n_k\}$ such that

$$\|\phi_{m_kn_k}\|_p\leq e^{\lambda_{m_k}(r_k+C_1)+\mu_{n_k}(r_k+C_2)}.$$

We now define

$$a_{mn}=\begin{cases}\dfrac{1}{\|\phi_{m_kn_k}\|_p}, & m=m_k,\ n=n_k\\ 0, & \text{otherwise.}\end{cases}$$

Then we see that

$$\limsup_{m+n\to\infty}\frac{\log|a_{mn}|+\lambda_m C_1+\mu_n C_2}{\lambda_m+\mu_n}\leq\lim_{k\to\infty} r_k=0,$$

i.e., $\sum_{m,n=1}^{\infty} a_{mn}\alpha_{mn}$ converges but $\|a_{m_kn_k}\phi_{m_kn_k}\|_p=1$ a contradiction.

In case when the functions $f(s_1, s_2)=\sum_{m,n=0}^{\infty} a_{mn} e^{\lambda_m s_1+\mu_n s_2}$ belonging to F are bounded on D and the topology $\mathscr{F}$ is weaker than sup norm topology

$$\|f\|_s = \sup_{(s_1, s_2)\in D} |f(s_1, s_2)|,$$

and if the condition (3.6) is replaced by a weaker condition (3.2) even then P remains continuous. In this direction we prove:

THEOREM 3.3. *Let F consist of bounded functions and the topology $\mathscr{F}$ be weaker than that determined by the sup norm topology on F. If the functions ϕ_{mn} belong to F and are uniformly continuous on D, then P is continuous linear mapping of E into F.*

PROOF. For $f\in E$ and δ, $-\infty<\delta<0$, we define

$$P_\delta f(s_1, s_2)=Pf(\delta+s_1, \delta+s_2),$$

then

$$P_\delta f(s_1, s_2)=\sum_{m,n=1}^{\infty} a_{mn}\phi_{mn}^{\delta}(s_1, s_2)$$

where

$$\phi_{mn}^{\delta}(s_1, s_2)=\phi_{mn}(\delta+s_1, \delta+s_2).$$

We note that if $\phi\in F$ and $\phi(s_1, s_2)=\sum_{m,n=1}^{\infty} c_{mn} e^{\lambda_m s_1+\mu_n s_2}$

$$\begin{aligned}\|\phi^{\delta}\|_s &= \sup_{(s_1,s_2)\in D} \sum_{m,n=1}^{\infty} c_{mn} e^{\lambda_m(\delta+s_1)+\mu_n(\delta+s_2)}\\ &\leq \sup_{\substack{\sigma_1<C_1\\ \sigma_2<C_2}} \sum_{m,n=1}^{\infty} |c_{mn}| e^{\lambda_m(\delta+\sigma_1)+\mu_n(\delta+\sigma_2)}\\ &=\|\phi : \delta+C_1, \ \delta+C_2\|.\end{aligned}$$

In view of the above observations and since the sup norm topology is stronger than $\mathscr{F}$, for each p, there exists K_p such that

$$\|\phi_{mn}^{\delta}\|_p\leq K_p\|\phi_{mn}^{\delta}\|_s\leq K_p\|\phi_{mn} : \delta+C_1, \ \delta+C_2\|.$$

Using (3.2) we get

$$\limsup_{m+n\to\infty}\frac{\log\|\phi_{mn}^{\delta}\|_p-\lambda_m C_1-\mu_n C_2}{\lambda_m+\mu_n}<0, \ \text{for } p\geq 1.$$

Hence for each δ, P_δ is continuous from E into F. Further since $\|P_\delta f\|_p\leq\|Pf\|_s$ for all $\delta(-\infty<\delta<0)$, therefore, P_δ is pointwise bounded. Moreover since ϕ_{mn}

are uniformly continuous on D, therefore,

$$\|P_{\delta}\delta_{mn}-P\delta_{mn}\|_s$$
$$=\sup_{(s_1,s_2)\in D}|\phi_{mn}(\delta+s_1,\ \delta+s_2)-\phi_{mn}(s_1,\ s_2)|\to 0$$

as $\delta\to 0$. Thus $P_{\delta}\to P$ on a total subset, applying Banach-Steinhaus Theorem [1, Theorem 18, p. 55], we see that P is continuous linear mapping of E into F.

4. Simultaneous automorphisms

We now consider two sequences $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ in E for which the difference functions

$$\phi_{mn}=\beta_{mn}-\alpha_{mn},\quad m,\ n\geq 1, \tag{4.1}$$

belong to F and satisfy (3.6). Since the topology $\mathscr{F}$ is stronger, for every $\sigma_1<C_1$, $\sigma_2<C_2$, we can find p and K such that

$$\|f:\sigma_1,\ \sigma_2\|\leq K\|f\|_p,\quad f\in F. \tag{4.2}$$

Using (3.6) and (4.2) we get (1.6) and hence there exists $\eta>0$ such that

$$\|\phi_{mn}:\sigma_1,\ \sigma_2\|<K_1e^{\lambda_m(C_1-\eta)+\mu_n(C_2-\eta)},\quad m,\ n\geq 1. \tag{4.3}$$

LEMMA 4.1. *Let $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ be two sequences in E for which the difference functions ϕ_{mn} $(m,\ n\geq 1)$ belong to F and satisfy (3.6), then the sequence $\{\beta_{mn}\}$ satisfies (1.6) if and only if $\{\alpha_{mn}\}$ does.*

PROOF. Suppose $\{\alpha_{mn}\}$ satisfies (1.6) then for arbitrary $\sigma_1<C_1$ and $\sigma_2<C_2$ there exist δ and K_2 such that

$$\|\alpha_{mn}:\sigma_1,\ \sigma_2\|\leq K_2e^{\lambda_m(C_1-\delta)+\mu_n(C_2-\delta)},\quad m,\ n\geq 1. \tag{4.4}$$

Hence using (4.3) and (4.4) we get

$$\|\beta_{mn}:\sigma_1,\ \sigma_2\|\leq K_1e^{\lambda_m(C_1-\eta)+\mu_n(C_2-\eta)}$$
$$+K_2e^{\lambda_m(C_1-\delta)+\mu_n(C_2-\delta)},$$

i.e.,
$$\limsup_{m+n\to\infty}\frac{\log\|\beta_{mn}:\sigma_1,\ \sigma_2\|-\lambda_mC_1-\mu_nC_2}{\lambda_m+\mu_n}\leq max(-\delta,\ -\eta)<0.$$

Since $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ are symmetrical in nature, they can be interchanged in the above arguments. Hence the result.

In the above Lemma 4.1, we can not replace (1.6) by (1.7) because if we consider $E=F$ and

$$-\alpha_{mn}(s_1, s_2)=\phi_{mn}(s_1, s_2)=e^{\lambda_m s_1+\mu_n s_2}, \ m, n\geq 1,$$

then $\{\alpha_{mn}\}$ satisfies (1.7) but $\{\beta_{mn}\}$ does not satisfy (1.7). But corresponding to the above lemma, we have the following:

LEMMA 4.2. *Let $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ be two sequences in E for which the functions ϕ_{mn}, $(m, n\geq 1)$, of* (4.1) *belong to F and satisfy*

$$\sup_{p\geq 1}\left\{\liminf_{m+n\to\infty}\frac{\log\|\phi_{mn}\|_p-\lambda_m C_1-\mu_n C_2}{\lambda_m+\mu_n}\right\}<0, \tag{4.5}$$

then the sequence $\{\beta_{mn}\}$ satisfies (1.7) *if and only if $\{\alpha_{mn}\}$ does.*

PROOF. In view of (4.5) there exists $\eta>0$ such that for every p

$$\|\phi_{mn}\|_p<e^{\lambda_m(C_1-\eta)+\mu_n(C_2-\eta)}$$

for sufficiently large $m+n$. Hence by (4.2) for every σ_1, σ_2 we have

$$\limsup_{m+n\to\infty}\frac{\log\|\phi_{mn}\,;\sigma_1,\ \sigma_2\|-\lambda_m C_1-\mu_n C_2}{\lambda_m+\mu_n}<-\eta<0,$$

or

$$\|\phi_{mn}\,;\sigma_1,\ \sigma_2\|<e^{\lambda_m(C_1-\eta)+\mu_n(C_2-\eta)}, \ m+n\geq N_1. \tag{4.6}$$

Further, if $\{\alpha_{mn}\}$ satisfies (1.7) we can find for δ, $0<\delta<\eta$, σ_1 and σ_2 close enough to C_1 and C_2 respectively, such that

$$\|\alpha_{mn}\,;\sigma_1,\ \sigma_2\|>e^{\lambda_m(-\delta+C_1)+\mu_n(-\delta+C_2)}, \ m+n\geq N_2. \tag{4.7}$$

Hence from (4.6) and (4.7) we get

$$\|\beta_{mn}\,;\sigma_1,\ \sigma_2\|\geq e^{\lambda_m(-\delta+C_1)+\mu_n(-\delta+C_2)}-e^{\lambda_m(-\eta+C_1)+\mu_n(-\eta+C_2)}$$

$$\geq e^{\lambda_m(-\delta+C_1)+\mu_n(-\delta+C_2)}[1-e^{(\delta-\mu)(\lambda_m+\mu_n)}], \ m+n\geq\max(N_1, N_2),$$

i.e.,

$$\liminf_{m+n\to\infty}\frac{\log\|\beta_{mn}\,;\sigma_1,\ \sigma_2\|-\lambda_m C_1-\mu_n C_2}{\lambda_m+\mu_n}>-\delta,$$

where σ_1, σ_2 are sufficiently close to C_1, C_2 respectively. Hence (1.7) is satisfied by $\{\beta_{mn}\}$ and the proof is complete due to symmetry of $\{\beta_{mn}\}$ and $\{\alpha_{mn}\}$.

Combining Lemmas 4.1 and 4.2, we get

THEOREM 4.1. *Let $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ be bases in E for which the functions ϕ_{mn}, $(m, n\geq 1)$ of* (4.1), *belong to F and satisfy* (4.5). *Then $\{\beta_{mn}\}$ is proper if and only if $\{\alpha_{mn}\}$ is proper.*

By an automorphism on the topological linear space (E, ξ) we mean a linear homeomorphic mapping of E onto itself. A simultaneous automorphism on E and F is then a mapping T such that T is an automorphism on (E, ξ) and T/F (T restricted to F) is an automorphism on $(F, \mathscr{T})$.

THEOREM 4.2. *Let* $\{\alpha_{mn}\}$ *and* $\{\beta_{mn}\}$ *be proper bases in* E *and let* T *be the endomorphism mapping* $\{\alpha_{mn}\}$ *onto* $\{\beta_{mn}\}$. *If the functions* ϕ_{mn}, $(m, n\geq 1)$, *of* (4.1) *belong to* F *and satisfy* (3.6), *then* T *is a simultaneous automorphism on* E *and* F.

PROOF. Let $f\in E$ and its expansion in the basis $\{\alpha_{mn}\}$ be given by

$$f=\sum_{m,n=1}^{\infty} a_{mn}\,\alpha_{mn}.$$

Then

$$Tf=\sum_{m,n=1}^{\infty} a_{mn}\,\beta_{mn}=\sum_{m,n=1}^{\infty} a_{mn}\,\alpha_{mn}+\sum_{m,n=1}^{\infty} a_{mn}\,\phi_{mn}.$$

Let P be as defined by (3.3) and I be the identity mapping. We see that $T=I+P$. Obviously I is a simultaneous automorphism on E and F. In view of Lemma 3.2, we see that P maps E into F continuously. Thus P also maps F into itself continuously and same is true for T.

Now we show that T maps F onto itself. Let $g\in F$ then $g=Tf$, for some $f\in E$.

$\Longrightarrow If=g-Pf$, is in F

$\Longrightarrow f\in F$, since I is a simultaneous automorphism on E and F. By Open Mapping Theorem [1, Theorem 2, p. 57] it follows that T is a simultaneous automorphism.

COROLLARY 4.1. *Let* $\{\alpha_{mn}\}$ *and* $\{\beta_{mn}\}$ *be bases in* E *for which the functions* ϕ_{mn} $(m, n\geq 1)$ *of* (4.1) *belong to* F *and satisfy* (4.5). *If one of the given bases is proper, then both are proper and the endomorphism* T *mapping* $\{\alpha_{mn}\}$ *onto* $\{\beta_{mn}\}$ *is a simultaneous automorphism on* E *and* F.

Department of Mathematics
and Astronomy,
Lucknow University,
Lucknow, India

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