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PROPERTIES OF RIEMANNIAN SPACE WITH PROJECTIVE CURVATURE TENSOR

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0. Introduction

Let us consider an $n(n>2)$ dimensional Riemannian space V_n with Weyl projective curvature tensor P^h_{ijk} given as follows ([2], [4])

$$(0.1) \qquad P^h_{ijk}=R^h_{ijk}-\frac{1}{(n-1)}(\delta^h_k R_{ij}-\delta^h_j R_{ik})$$

$$(0.2) \qquad P^h_{ijk,\,l}=0.$$

Then V_n is called the *projective symmetric space*. Where (,) denotes the covariant differential.

The space V_n is called *Projective recurrent* or *Projective birecurrent* respectively if it satisfies [4]

$$(0.3) \qquad P^h_{ijk,\,l}=K_l P^h_{ijk},$$

or

$$(0.4) \qquad P^h_{ijk,\,lm}=a_{lm}P^h_{ijk}$$

where K_l and a_{lm} are called *vector* and *tensor of recurrence* resp. Gupta [2] has shown that in a projective space $(n>2)$ the scalar curvature R is constant, that is

$$(0.5) \qquad R_{,l}=0$$

so that in an Einstein space, we have

$$(0.6) \qquad R_{ij,l}=0,$$

since $\qquad R_{ij}=\frac{R}{n}g_{ij}.$

Contracting for h and k respectively in (0.1), we get

$$(0.7) \qquad P^a_{ija}=P^a_{iaj}=P^a_{aij}=0.$$

From (0.1), we also have

$$(0.8) \qquad P_{hk}=g^{ij}P_{hijk}=\frac{n}{(n-1)}\left(R_{hk}-\frac{R}{n}g_{hk}\right).$$

Let v^h be any arbitrary vector field satisfying [1]

(0.9) $$v^h_{,l}=c\delta^h_l+v^h B_l,$$

then v^h is called *concircular vector field,* where B_l and C are some scalar fields.

1. Projective recurrent and birecurrent space

Differentiating (0.3) covariantly and using (0.3), we get

(1.1) $$a_{mn}=K_{m,n}+K_m K_n.$$

Let

(1.2) $$A_{mn}=a_{mn}-a_{nm},$$

which in view of (1.1), yields

(1.3) $$A_{mn}=K_{m,n}-K_{n,m}.$$

It can be easily verified that a vector of recurrence is gradient then the 2nd order recurrent tensor is symmetric.

THEOREM 1.2. *If the Projective recurrent space P_n is an Einstein space, then β_m (the vector of recurrence) is covariantly constant.*

PROOF. Differentiating (0.1) covariantly then using Bianchi identity and (0.6), we get

(1.4) $$P^h_{ijk,m}+P^h_{ikm,j}+P^h_{imj,k}=0.$$

Again differentiating (1.4), covariantly and making use of (0.4), we obtain

(1.5) $$a_{mt}P^h_{ijk}+a_{jt}P^h_{ikm}+a_{kt}P^h_{imj}=0.$$

Let

(1.6) $$a_{mt}V^t \overset{\text{def}}{=} \beta_m.$$

From (1.5) and (1.6), we have

(1.7) $$\beta_m P^h_{ijk}+\beta_j P^h_{ikm}+\beta_k P^h_{imj}=0.$$

Comparing (1.4) and (1.7) and in view of (0.3), we say that β_m is a *vector of recurrence.*

Differentiating (1.7) and with the help of (0.3) and (1.7), we can get a relation as follows

(1.8) $$\beta_{m,n}P^h_{ijk}+\beta_{j,n}P^h_{ikm}+\beta_{k,n}P^h_{imj}=0.$$

Contracting for h and k or h and j in (1.8) and using (0.7), we find $\beta_{r,n}P^r_{imj}=0.$

Since $P^r_{imj} \neq 0$, therefore $\beta_{r,n}=0$.

THEOREM 1.3. *In a projective recurrent space let the vector of recurrence be a gradient then it is an Einstein space.*

PROOF. From (0.1) and (0.6), we have

$$(1.9)\qquad (P_{hijk,lm}-P_{hijk,ml})+(P_{jklm,hi}-P_{jklm,ih})+(P_{lmhi,jk}-P_{lmhi,kj})=0$$

where we have used the Ricci identity so that

$$(1.10)\qquad (R_{hijk,lm}-R_{hijk,ml})+(R_{jklm,hi}-R_{jklm,ih})+(R_{lmhi,jk}-R_{lmhi,kj})=0.$$

From (0.4), (1.2) and (1.9), we obtain

$$(1.11)\qquad A_{lm}P_{hijk}+A_{hi}P_{jklm}+A_{jk}P_{lmhi}=0$$

since in an Einstein space $P_{hijk}=P_{jkhi}$.

We give here a lemma due to Walker [10] as follows:

LEMMA 1. *If $a_{\alpha\beta}$ and b_α are numbers satisfying*

$$a_{\alpha\beta}=a_{\beta\alpha} \text{ and } a_{\alpha\beta}\, b_\gamma+a_{\beta\gamma}\, b_\alpha+a_{\gamma\alpha}\, b_\beta=0,$$

then either all b_α are zero or all $a_{\alpha\beta}$ are zero.

Therefore all $A_{lm}=0$, because $P_{hijk}\neq 0$. Hence the theorem follows.

THEOREM 1.4. *If the Projective space V_n be Ricci-recurrent and K_l is gradient then K^*_l is also gradient.*

PROOF. In a Ricci-recurrent space, we have [7]

$$(1.12)\qquad R_{ij,l}=K^*_l\, R_{ij}.$$

Let us assume that

$$(1.13)\qquad \Pi_{ij}=\left(R_{ij}-\frac{Rg_{ij}}{n-2}\right).$$

From (1.12) and (1.13), we get

$$(1.14)\qquad \Pi_{ij,l}=K^*_l\Pi_{ij}.$$

In view of (0.1) and (1.13), we have

$$(1.15)\qquad D^{*h}_{ijk}\overset{\text{def}}{=}R^h_{ijk}-P^h_{ijk}=\frac{1}{(n-1)}\left\{\delta^h_j\Pi_{ik}-\delta^h_k\Pi_{ij}+R(\delta^h_j\, g_{ik}-\delta^h_k g_{ij})\right\}$$

which in view of (1.12) and (1.13), yields

$$(1.16)\qquad D^h_{ijk,l}=K^*_l D^h_{ijk}.$$

Using (0.4), (1.2), (1.10), (1.15), (1.16) and also the fact that K_l is gradient, we can obtain

(1.17) $$A^*_{lm}D_{hijk}+A^*_{ih}D_{jklm}+A^*_{jk}D_{lmhi}=0,$$

where $$A^*_{lm}=K^*_{l,m}-K^*_{m,l}$$

since $D_{hijk}=D_{jkhi}$, then we have all $A^*_{lm}=0$, because all D_{hijk} are not zero. Hence the theorem follows.

2. Projective symmetric space

THEOREM 2.1. *In a projetive symmetric space* $(n>2)$ $R^j_{k,j}=0$.

PROOF. Let us consider a tensor V^h_{ijkl}, such that [9]

(2.1) $$V^h_{ijkl}=R^h_{ijk,l}+R^h_{kil,j}+R^h_{lkj,i}+R^h_{jli,k}=0$$

which on contraction yields

(2.2) $$V^h_{ijkh}=R^h_{ijk,h}+R_{ki,j}-R_{ji,k}=0.$$

Multiplying (2.2) by g^{ij} and summing on i and j, we find

(2.3) $$g^{ij}V^h_{ijkh}=2R^j_{k,j}-R_{,k}.$$

From (0.1) and (0.2), we get

(2.4) $$R^h_{ijk,l}=\frac{\delta^h_kR_{ij,l}-\delta^h_jR_{ik,l}}{(n-1)}.$$

Substituting (2.4) in (2.1) and contracting the resulting equation on h and l and then multiplying by g^{ij}, we get

(2.5) $$g^{ij}V^h_{ijkh}=\frac{n-2}{n-1}(R^j_{k,j}-R_{,k}),$$

which in view of (2.3) reduces to

(2.6) $$R^j_{k,j}=\frac{R_{,k}}{n}.$$

Since in a projective symmetric space $(n>2)$, we have R=constant, therefore the theorem follows.

COROLLARY 1. *In a projective symmetric space*

(2.7) $$P_{ij,k}+P_{jk,i}+P_{ki,j}=0,\ \textit{if}$$

(2.8) $$R_{ij,k}+R_{jk,i}+R_{ki,j}=0\ \textit{ is satisfied.}$$

PROOF. From (0.2), we get [5]

(2.9) $$P_{hk}=\frac{n}{n-1}\left(R_{hk}-\frac{Rg_{hk}}{n}\right)$$

(2.9), in view of (2.8) reduces to (2.7). Hence the corollary follows.

COROLLARY 2. *If the projective symmetric space* ($n>2$) *admits a concircular vector field, then the following relations hold.*

(2.10)a $$CP_{lijk}+g_{ki}T_{jl}-g_{ij}T_{kl}$$
$$=\frac{1}{(n-1)}\{Cg_{jl}P_{ik}-Cg_{kl}R_{ij}+v_jR_{ik,l}-v_kR_{ij,l}\},$$

(2.10)b $$CP_{lk}+(1-n)T_{kl}=C\frac{(R_{lk}-Rg_{lk})}{n-1}+v_j\frac{R^j_{k,l}}{(n-1)},$$

and

(2.10)c $$T_{ji}-T_{rs}g^{rs}=-CR_{ij}-\frac{v^lR_{ij,l}}{(n-1)}$$

where

(2.11)a $$T_{jk}=D_jB_k-D_{j,k},$$

and

(2.11)b $$D_j=C_j-CB_j$$

also $$C_j=C_{,j}.$$

PROOF. From (0.9), we can write

(2.12) $$v_{h,l}=Cg_{hl}+v_hB_l.$$

Differentiating (2.12) covariantly and making use of (2.10)—(2.12), we get

(2.13) $$v_sR^s_{hij}=D_ig_{hj}-D_jg_{hi}.$$

From (0.1) and (2.13), we obtain

(2.14) $$v_sP^s_{ijk}=D_jg_{ki}-D_kg_{ji}-\frac{1}{(n-1)}(v_kR_{ij}-v_jR_{ik}).$$

Differentiating (2.14) covariantly and taking account of (2.11)—(2.14), we find that

$$CP_{lijk}+g_{ki}T_{jl}-g_{ji}T_{kl}=\frac{1}{(n-1)}\left\{C(g_{jl}R_{ik}-g_{kl}R_{ij})+(v_jR_{ik,l}-v_kR_{ij,l})\right\}$$

which on multiplication with g^{ij} and g^{lk}, yields (2.10)b, c respectively.

THEOREM 2.2. *In a projective symmetric space* ($n>2$) *with a concircular vector field* v^h, *we have*

(2.15) $$v^rv^sP_{ks,r}=v^rv^sP_{rs,k}=0$$ *provided the relation* (2.9) *holds.*

PROOF. Contracting (2.10)b by g^{kl} and using (0.5), (0.7), we obtain

(2.16) $$T_{rs}g^{rs}=\frac{CR}{n-1}.$$

In view of (2.10)b, (2.16), we obtain that T_{ij} is symmetric w.r.t. i and j. Interchanging l and k in (2.10)b and then subtracting the resulting equation from (2.10)b, where (2.7) is also used, yields

(2.17) $$v^jP_{jk.l}=v^jP_{jl.k},$$

or Equivalently

(2.18) $$v^jv^k\cdot P_{jk,l}=v^jv^kP_{jl,k}.$$

Transvecting (2.7) with v^j and putting $j=s$, we get

(2.19) $$v^sP_{ik,s}=-2v^sP_{is,k},$$

from which on multiplication by v^i and changing the dummy indices, we get

(2.20) $$v^rv^sP_{rk,s}=-2v^rv^sP_{rs,k}.$$

From (2.18) and (2.20), we obtain (2.15).

THEOREM 2.3. *If the projective symmetric space is an Einstein space, we have*

(2.21)a $$T_{ji}=\frac{CRg_{ji}}{n(n-1)}$$

and

(2.21)b *either* $C=0$ *or* $P_{hijk}=0$.

PROOF. From (0.5), (0.6), (2.10) and (2.16), we obtain (2.21)a. Substituting (0.5), (0.6) and (2.21)a in (2.10)a, we get (2.21)b.

3. Projective Veblen identity

Let us define as follows

(3.1) $$U^h_{ijkl}=P^h_{ijk,l}+P^h_{kil,j}+P^h_{ikj,l}+P^h_{jli,k}.$$

From (0.1), (2.1) and (3.1), we get

(3.2) $$U^h_{ijkl}=V^h_{ijkl}+\frac{1}{(n-1)}\{\delta^h_j(R_{ik,l}-R_{lk,i})+\delta^h_k(R_{jl,i}-R_{ij,l})$$
$$+\delta^h_l(R_{ij,k}-R_{ik,j})+\delta^h_i(R_{lk,j}-R_{lj,k})\}.$$

On contraction for h and l and in view of (2.2), we find

(3.3) $$U^h_{ijkh}=V^h_{ijkh}+\frac{(n-2)}{(n-1)}(R_{ij,k}-R_{ik,j})=P^h_{ijk,h}.$$

Thus

$$R_{ij,k}-R_{ik,j}=\frac{n-1}{n-2}(P^h_{ijk,h}-V^h_{ijkh}). \tag{3.4}$$

In consequence of (3.2) and (3.4), we have

$$\begin{aligned} W^{*h}_{ijkl} &\overset{\text{def}}{=} P^h_{ijk,l}+P^h_{kil,j}+P^h_{lkj,i}+P^h_{jli,k} \\ &-\frac{g^{hm}}{(n-2)}\{g_{jm}P^p_{kli,p}+g_{km}P^p_{jli,p}+g_{lm}P^p_{ijk,p}+g_{im}P^p_{ljk,p}\} \\ &=V^h_{ijkl}-\frac{g^{hm}}{(n-2)}\{g_{jm}V_{kli}+g_{km}V_{jli}+g_{lm}V_{ijk}+g_{im}V_{lkj}\}. \end{aligned} \tag{3.5}$$

The r.h.s. of (3.5) in view of (2.1) and (2.2), vanishes. On the analogy of conformal Veblen identity [9], we call W^{*h}_{ijkl} the *projective Veblen tensor* and equation (3.5) *projective Veblen identity.*

THEOREM 3.1. *Projective Veblen identity in a Riemannian space and the Einstein space are identical.*

PROOF. Substituting (0.6) in (3.2), we get

$$U^h_{ijkl}=V^h_{ijkl}. \tag{3.6}$$

From (3.1), (3.5) and (3.6), we obtain

$$\begin{aligned} g^{hm}(P^p_{kli,p}g_{jm}+P^p_{jli,p}g_{km}+P^p_{ijk,p}g_{lm}+P^p_{lkj,p}g_{im} \\ -g_{jm}V_{kli}-g_{km}V_{jli}-g_{lm}V_{ijk}-g_{im}V_{lkj})=0. \end{aligned} \tag{3.7}$$

In a Riemannian space ordinary Veblen identity is satisfied therefore (3.6) holds. Thus we have

$$U^h_{ijkl}=0. \tag{3.8}$$

Hence the theorem follows.

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