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THE OPERATOR $T_{k,q}$ AND CHARACTERIZATION OF A CLASS OF POLYNOMIALS BY THE GENERALIZED RODRIGUE'S FORMULA

By C.M. Joshi and M.L. Prajapat

1. Introduction

Employing the operator x^2D where $D=\frac{d}{dx}$ Chak [2] defined the generalized Laguerre polynomials by means of

$$L_n^{(\alpha)}(x)=x^{-\alpha-n-1}e^x(x^2D)^n(x^{\alpha+1}e^{-x}) \tag{1.1}$$

Later, Al-Salam [1] characterized these polynomials in terms of the operator $\theta=x(1+xD)$ and proved that

$$\theta^n x^\alpha e^{-x}=x^{\alpha+n}e^{-x}n!\,L_n^{(\alpha)}(x) \tag{1.2}$$

Recently, Mittal [7] observed that relations (1.1) and (1.2) can, in fact, be derived from a more general operational representation. To this end he considered the operator $T_k=x(k+xD)$, k being constant and showed that the polynomial set $\{T_{\nu n}^{(\alpha)}(x)\,|\,n=0, 1, 2, \cdots\}$ [6] where

$$T_{\nu n}^{(\alpha)}(x)=\frac{1}{n!}x^{-\alpha}e^{p_\nu(x)}D^n[x^{\alpha+n}e^{-p_\nu(x)}] \tag{1.3}$$

$p_\nu(x)$ is a polynomial in x of degree r, admit the relationship

$$T_{\nu n}^{(\alpha+k-1)}(x)=\frac{1}{n!}x^{-\alpha-n}e^{p_\nu(x)}T_k^n[x^\alpha e^{-p_\nu(x)}] \tag{1.4}$$

in terms of the operator T_k.

The question, that attracted our attention, was if these aforementioned characterizations could be unified. This led us [5] to define the operator $T_{k,q}=x^q(k+xD)$ and the introduction of the polynomial set in the form

$$M_{\nu n}^{(\alpha)}(x, k, q)=\frac{1}{n!}x^{-\alpha-nq}e^{p_\nu(x)}T_{k,q}^n[x^\alpha e^{-p_\nu(x)}] \tag{1.5}$$

where $p_\nu(x)$ is a polynomial in x of degree r and k and q are constants.

In so far as the generality is concerned, obviously the definition is quite general. Indeed, it provided a direct generalization of all the known generalizations of classical Laguerre polynomials for which one may refer to the work of

Chatterjea [3] and Singh and Srivastava [9]. Yet the definition is limited, since it is not possible to carry over a number of well known properties to the generalized case. This limitation is, to a great extent, overcome by introducing the polynomial set $\{M_n^{(\alpha)}(x, r, p, k, q)\}, n=0, 1, 2, \cdots\}$,

$$M_n^{(\alpha)}(x, r, p, k, q) = \frac{1}{n!} x^{-\alpha-nq} e^{px^r} T_{k,q}^n (x^\alpha e^{-px^r}) \tag{1.6}$$

where p, r, k and q are constants and assume integral values.

For $k=0$, one would obtain the polynomial set $\{G_n^{(\alpha)}(x, r, p, q)\}, n=0, 1, 2, \cdots\}$ considered earlier by Srivastava and Singhal [10]. It may be noted here that the polynomial set $\{G_n^{(\alpha)}(x, r, p, q)\}$ can not be said to give a direct generalization of the classical Hermite or the generalized Hermite polynomials of Gould and Hopper [4], since

$$G_n^{(0)}(x, 2, 1, -1) = \frac{(-x)^n}{n!} H_n(x),$$

and

$$G_n^{(\alpha)}(x, r, p, -1) = \frac{(-x)^n}{n!} H_n^r(x, \alpha, p).$$

The question, therefore, naturally arises if there does in fact exist a unified representation for the two classes of seemingly alike polynomials, the Laguerre and Hermite polynomials. Interestingly, the answer is in the affirmative and will be presented in detail in a subsequent communication.

2. The operator $T_{k,q}$

We [5] have defined the operator $T_{k,q}$ as

$$T_{k,q} \equiv x^q (k+xD), \text{ where } D \equiv \frac{d}{dx}.$$

Listed below are some of the properties that we shall require in our investigations:

$$T_{k,q}^n (x^{\alpha+m}) = q^n \left(\frac{\alpha+m+k}{q} \right)_n x^{\alpha+m+nq} \tag{2.1}$$

where as usual $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$, $n \geqslant 1$, $(\alpha)_0 = 1$.

$$F(T_{k,q}) \{x^\alpha f(x)\} = x^\alpha F(T_{k,q} + x^q \alpha) f(x) \tag{2.2}$$

$$F(T_{k,q}) \{e^{g(x)} f(x)\} = e^{g(x)} F\{T_{k,q} + x^{q+1} g'(x)\} f(x) \tag{2.3}$$

$$(T_{k,q})^n (xuv) = x \sum_{m=0}^{n} \binom{n}{m} (T_{k,q}^{n-m} v)(T_{1,q}^m u) \tag{2.4}$$

where $T_{1,q}=x^q(1+xD)$

In particular,

$$(T_{k,q})^n(uv)=\sum_{m=0}^{n}\binom{n}{m}(T_{k,q}^{n-m}v)(T_q^m u),\ T_q=x^{q+1}D \tag{2.5}$$

$$e^{tT_{k,q}}[x^\alpha f(x)]=\frac{x^\alpha}{(1-x^q qt)^{\frac{\alpha+k}{q}}}f\left[\frac{x}{(1-x^q qt)^{1/q}}\right] \tag{2.6}$$

$${}_\lambda F_\mu\left[\begin{matrix}(a_\lambda);\\(b_\mu);\end{matrix}\ tT_{k,q}\right]x^\alpha e^{px^r}=\sum_{j=0}^{\infty}\frac{(p)^j}{j!}x^{\alpha+rj}\ {}_{\lambda+1}F_\mu\left[\begin{matrix}(a_\lambda),\ \left(\frac{\alpha+rj+k}{q}\right);\\(b_\mu);\end{matrix}\ x^q qt\right] \tag{2.7}$$

where (a_λ) stands for the sequence of λ parameters namely $a_1, a_2, \cdots a_\lambda$ with similar interpretation for (b_μ).

In particular

$${}_0F_1\left[-;\ \frac{\alpha+k}{q};\ tT_{k,q}\right]x^\alpha e^{-px^r}=x^\alpha\sum_{m=0}^{\infty}\frac{(-px^r)^m}{m!}\ {}_1F_1\left[\frac{\alpha+k+mr}{q};\ \frac{\alpha+k}{q};x^q qt\right] \tag{2.8}$$

In addition, note also that

$$\prod_{j=0}^{n-1}(\delta+\alpha+k-prx^r+jq).1=n!\ M_n^{(\alpha)}(x,r,p,k,q) \tag{2.9}$$

which can be put in an equivalent form

$$M_n^{(\alpha)}(x,r,p,k,q)=\frac{q^n}{n!}e^{px^r}\left(\frac{\delta+\alpha+k}{q}\right)_n e^{-px^r},\ \delta=xD, \tag{2.10}$$

and suggests the elegant operational relationship

$$x^{-nq}T_{k,q}^n\equiv(\delta+k)(\delta+k+q)\cdots(\delta+k+\overline{n-1}q). \tag{2.11}$$

3. The explicit form

It follows from (1.6) and (2.1) that

$$M_n^{(\alpha)}(x,r,p,k,q)=\frac{q^n}{n!}\sum_{m=0}^{\infty}\frac{(-px^r)^m}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left(\frac{\alpha+rj+k}{q}\right)_n \tag{3.1}$$

This can be put in the form

$$M_n^{(\alpha)}(x,r,p,k,q)=\frac{a^n}{n!}e^{px^r}\ (a)_n\ y \tag{3.2}$$

where $y=\sum_{j=0}^{\infty} \frac{(-p)^j}{j!} \frac{(a+n)_{mj}}{(a)_{mj}} x^{rj}$, $\frac{\alpha+k}{q}=a$,

$\frac{r}{q}=m$, m being a positive integer.

Now, if

$$\Delta_{\alpha, r} f(\alpha)=f(\alpha+r)-f(\alpha)$$

so that

$$\Delta_{\alpha, r}^{m} f(\alpha)=\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j} f(\alpha+rj), \tag{3.3}$$

it becomes obvious that the inner series in (3.1) can be expressed as the m^{th} difference of a polynomial of degree n in α which vanishes for $m>n$, such that

$$M_n^{(\alpha)}(x, r, p, k, q)=\frac{q^n}{n!}\sum_{m=0}^{n}\frac{(-px^r)^m}{m!}\Delta_{\alpha+k, r}^{m}\left(\frac{\alpha+k}{q}\right)_n \tag{3.4}$$

This suggests on the one hand that $M_n^{(\alpha)}(x, r, p, k, q)$ is a polynomial of degree n in x^r, since one could write

$$M_n^{(\alpha)}(x, r, p, k, q)=\frac{q^n}{n!}\exp\left[-px^r\Delta_{\alpha+k, r}\right]\left(\frac{\alpha+k}{q}\right)_n \tag{3.5}$$

whereas in view of the definition and the formula (2.5) $M_n^{(\alpha)}(x, r, p, k, q)$ can be expressed as a polynomial of degree n in α in the form

$$M_n^{(\alpha)}(x, r, p, k, q)=\sum_{m=0}^{n}\frac{q^m}{m!}\left(\frac{\alpha}{q}\right)_m M_{n-m}^{(0)}(x, r, p, k, q) \tag{3.6}$$

4. The differential equation

Assuming that $\frac{r}{q}=m$, a positive integer and employing the operator $T_{k,q}$ possessing the property that

$$x^{-q}T_{k,q}x^n=(n+k)x^n,$$

in view of (3.2), we obtain

$$[x^{-q}T_{k,q}-k]\left[\frac{m}{r}(x^{-q}T_{k,q}-k)+a-m\right]_m y$$

$$=r\sum_{j=0}^{\infty}\frac{(-p)^{j+1}(a+n)_{mj+m}x^{rj+r}}{j!(a)_{mj}}$$

$$=-prx^r\left[\frac{m}{r}\{x^{-q}T_{k,q}-k\}+a+n\right]_m y.$$

This shows that the polynomials satisfy the differential equation

$$\Big[(x^{-q}T_{k,q}-k-prx^{r})\Big\{\frac{m}{r}(x^{-q}T_{k,q}-k)-prx^{r}+a-m\Big\}_{m}$$
$$+prx^{r}\Big\{\frac{m}{r}(x^{-q}T_{k,q}-k)-prx^{r}+a+n\Big\}_{m}\Big]M_{n}^{(\alpha)}(x,r,p,k,q)=0 \tag{4.1}$$

Or, since (2.11) holds, we have alternatively

$$\Big[(\delta-prx^{r})\Big(\frac{m}{r}\delta-pmx^{r}+a-m\Big)_{m}+prx^{r}\Big(\frac{m}{r}\delta-pmx^{r}$$
$$+a+n\Big)_{m}\Big]M_{n}^{(\alpha)}(x,r,p,k,q)=0, \tag{4.2}$$

giving rise to the product form

$$\Big[(\delta-prx^{r})\prod_{j=1}^{m}(\delta-prx^{r}+\alpha+k-r+jq-q)$$
$$+prx^{r}\prod_{j=1}^{m}(\delta-prx^{r}+\alpha+k+nq+jq-q)\Big]M_{n}^{(\alpha)}(x,r,p,k,q)=0. \tag{4.3}$$

If, however, $\frac{r}{q}=-m$, where m is a positive integer, the differential equation assumes the form

$$\Big[(\delta-prx^{r})\prod_{j=1}^{m}(\delta-prx^{r}-\alpha-k-r-nq+jq)$$
$$+prx^{r}\prod_{j=1}^{m}(\delta-prx^{r}-\alpha-k+jq)\Big]M_{n}^{(\alpha)}(x,r,p,k,q)=0 \tag{4.4}$$

5. Generating functions

For the sake of brevity, if we assume that $[(a_\lambda)]_n$ and $[(b_\mu)]_n$ stand for $\prod_{j=1}^{\lambda}(a_j)_n$ and $\prod_{j=1}^{\mu}(b_j)_n$ respectively, then it follows from (1.6) that

$$\sum_{n=0}^{\infty}\frac{[(a_\lambda)]_n}{[(a_\mu)]_n}M_{n}^{(\alpha)}(x,r,p,k,q)\,t^n$$
$$=e^{px^r}x^{-\alpha}\,{}_{\lambda}F_{\mu}\Big[\begin{matrix}(a_\lambda);\\(b_\mu);\end{matrix}\;x^{-q}tT_{k,q}\Big]x^{\alpha}e^{-px^r}$$

Therefore, by virtue of (2.7), one obtains the generating relation

$$\sum_{n=0}^{\infty}\frac{[(a_\lambda)]_n}{[(b_\mu)]_n}M_{n}^{(\alpha)}(x,r,p,k,q)\,t^n$$
$$=e^{px^r}\sum_{n=0}^{\infty}\frac{(-px^r)^n}{n!}\,{}_{\lambda+1}F_{\mu}\Big[\begin{matrix}(a_\lambda),\Big(\dfrac{\alpha+k+nr}{q}\Big);\\(b_\mu);\end{matrix}\;qt\Big] \tag{5.1}$$

In particular, for $\lambda=\mu$, $a_j=b_j$, $j=1, 2, \cdots, \lambda$ (or μ), (5.1) reduces to

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x, r, p, k, q)\, t^n=(1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r\{1-(1-qt)^{-\frac{r}{q}}\}\right] \tag{5.2}$$

Next in (3.1), if we replace α by α-nq, multiply both the sides by t^n and sum for $n\geqslant 0$, we obtain

$$\sum_{n=0}^{\infty} M_n^{(\alpha-nq)}(x, r, p, k, q)\, t^n=(1+qt)^{\frac{\alpha+k-q}{q}} \exp\left[px^r\{1-(1+qt)^{\frac{r}{q}}\}\right] \tag{5.3}$$

Multiplication of this by $x^{\alpha}\, e^{-px^r}$ and then operation by $T_{k,q}^m$ yields

$$\sum_{n=0}^{\infty} \binom{m+n}{n} M_{m+n}^{(\alpha-nq)}(x, r, p, k, q)\, t^m=(1+qt)^{\frac{\alpha+k-q}{q}}\cdot\exp\left[px^r\{1-(1+qt)^{\frac{r}{q}}\}\right]$$

$$\cdot M_n^{(\alpha)}\left[\frac{x}{(1+qt)^{-\frac{1}{q}}}, r, p, k, q\right] \tag{5.4}$$

On the other hand, since

$$\sum_{n=0}^{\infty} \binom{m+n}{n} M_{m+n}^{(\alpha)}(x, r, p, k, q)\, t^n$$

$$=x^{-\alpha-mq}\, e^{px^r}\, e^{x^{-q}t}\, T_{k,q}[x^{\alpha+mq}\, e^{-px^r}\, M_m^{(\alpha)}(x, r, p, k, q)], \tag{5.5}$$

by an appeal to (2.6), we get

$$\sum_{n=0}^{\infty} \binom{m+n}{n} M_{m+n}^{(\alpha)}(x, r, p, k, q)\, t^n$$

$$=(1-qt)^{-\left(\frac{\alpha+k}{q}\right)-m}\cdot\exp\left[px^r\{1-(1-qt)^{-\frac{r}{q}}\}\right] M_m^{(\alpha)}\left[\frac{x}{(1-qt)^{\frac{1}{q}}}, r, p, k, q\right] \tag{5.6}$$

where, $m=0, 1, 2, \cdots$.

Interestingly, for $m=0$, (5,4) and (5.6) reduce to (5.3) and (5.2) respectively. Again, by definition

$$\sum_{n=0}^{\infty} \frac{(x^q t)^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x, r, p, k, q)=x^{-\alpha}\, e^{px^r}\, {}_0F_1\left[-;\ \frac{\alpha+k}{q};tT_{k,q}\right] x^{\alpha}\, e^{-px^r} \tag{5.7}$$

Hence by making use of (2.8), it simplifies to

$$\sum_{n=0}^{\infty} \frac{t^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x, r, p, k, q)=e^{px^r+qt} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\, {}_1F_1\left[-\frac{mr}{q};\ \frac{\alpha+k}{q};\ -qt\right] \tag{5.8}$$

We alse notice from (2.8) that if $\frac{r}{q}=s$, s is a positive integer, then

$$ {}_0F_1\left[-;\ \frac{\alpha+k}{q};\ tT_{k,q}\right] x^{\alpha} e^{-px^r} $$

$$ = x^{\alpha} \sum_{n=0}^{\infty} \frac{(x^q qt)^n}{n!}\ {}_SF_S\left[\begin{matrix}\Delta\left(S,\ \dfrac{\alpha+k+nq}{q}\right); \\ \Delta\left(S,\ \dfrac{\alpha+k}{q}\right);\end{matrix}\ -px^r\right] \tag{5.9} $$

where $\Delta(S, \alpha)$ stands for the set of s parameters

$$ \frac{\alpha}{S},\ \frac{\alpha+1}{S},\ \ldots,\ \frac{\alpha+S-1}{S}. $$

This on comparison with (5.7) yields another form of the explicit formula

$$ M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q) = \frac{q^n\left(\frac{\alpha+k}{q}\right)_n}{n!}\ e^{px^r}\ {}_SF_S\left[\begin{matrix}\Delta\left(S,\ \dfrac{\alpha+k+nq}{q}\right); \\ \Delta\left(S,\ \dfrac{\alpha+k}{q}\right);\end{matrix}\ -px^r\right] \tag{5.10} $$

Next, by making an appeal to (2.10), we obtain

$$ \sum_{n=0}^{\infty} \frac{\left(\frac{c}{q}\right)_m}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)\, t^n $$

$$ = e^{px^r}(1-qt)^{-\frac{c}{q}} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\ {}_2F_1\left[-\frac{m}{q},\ \frac{c}{q};\ \frac{\alpha+k}{q};\ \frac{qt}{qt-1}\right] \tag{5.11} $$

If we put $\lambda=\mu=1$, $a_1=\frac{c}{q}$ and ${}_1=\frac{\alpha+k}{q}$, then (5.1) also leads us to (5.11).

6. Some applications of generating functions

Observe that the generating relations (5.2) and (5.3) on comparison, yield the recursion formula

$$ M_n^{(\alpha)}(x,\ r,\ p,\ k,\ -q) = M_n^{(\alpha-nq)}(x,\ r,\ p,\ k,\ q) + qM_{n-1}^{(\alpha-nq+q)}(x, r, p, k, q) \tag{6.1} $$

Two other immediate consequences of (5.2) are

$$ DM_n^{(\alpha)}(x,\ r,\ p,\ k,\ q) = prx^{r-1}\left[M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q) - M_n^{(\alpha+r)}(x, r, p, k, q)\right] \tag{6.2} $$

and

$$ (\alpha+k-q)M_n^{(\alpha+k)}(x,\ r,\ p,\ k,\ q) = prx^r M_n^{(\alpha+k+r)}(x,\ r,\ p,\ k,\ q) $$

$$ +(n+1)\, M_{n+1}^{(\alpha-q)}(x,\ r,\ p,\ k,\ q) \tag{6.3} $$

Further, in view of the generating relation (5.2), we readily establish the following multiplication, addition and the summation formulas:

$$M_n^{(\alpha)}(x,\ r,\ mp,\ k,\ q)=M_n^{(\alpha)}(m^{\frac{1}{r}}x,\ r,\ p,\ k,\ q) \tag{6.4}$$

$$M_n^{(\sum_{s=1}^{m}\alpha_s)}\left[x,\ r,\sum_{s=1}^{m}f_s\ mk,\ q\right]=\sum_{i_1+\cdots+i_m=n}\prod_{j=1}^{m}M_{i_j}^{(\alpha_j)}(x,\ r.\ p_j,\ k,\ q) \tag{6.5}$$

$$M_n^{\left(\sum_{s=1}^{m}\alpha_s\right)}\left[\left(\sum_{s=1}^{m}x_s\right)^{\frac{1}{r}},\ r,\ p,\ mk,\ q\right]=\sum_{i_1+\cdots+i_m=n}\prod_{j=1}^{m}M_{i_j}^{(\alpha_j)}\left[(x_j^{\frac{1}{r}})\ r,\ p,\ k,\ q\right] \tag{6.6}$$

and

$$M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)=\sum_{m=0}^{n}\frac{\left(\frac{\alpha-\beta}{q}\right)_m}{m!}\ q^m M_{n-m}^{(\beta)}(x,\ r,\ p,\ k,\ q) \tag{6.7}$$

Note also from definition that

$$T_{k,q}^{m}\left[e^{-px^r}x^{\alpha+nq}M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)\right]$$
$$=\frac{(m+n)!}{n!}\ x^{\alpha+\overline{m+n}\,q}\ \ e^{-px^r}M_{m+n}^{(\alpha)}(x,\ r,\ p,\ k,\ q) \tag{6.8}$$

Therefore by an appeal to (2.2) and (2.3), one obtains

$$[T_{k,q}+(\alpha+nq)\,x^q-prx^{r+q}]^m\ M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)$$
$$=\frac{(m+n)!}{n!}x^{mq}\ M_{m+n}^{(\alpha)}(x,\ r,\ p,\ k,\ q). \tag{6.9}$$

From $m=1$, it reduces to

$$[xD+\alpha+k+nq-prx^r]\ M_n^{(\alpha)}(x,r,p,k,q)=(n+1)M_{n+1}^{(\alpha)}(x,r,p,k,q) \tag{6.10}$$

Obviously, elimination of $DM_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)$ between (6.2) and (6.10) would lead to the recurrence relation

$$(n+1)\,M_{n+1}^{(\alpha)}(x,r,p,k,q)=(\alpha+k+nq)M_n^{(\alpha)}(x,r,p,k,q)$$
$$-prx^r M_n^{(\alpha+r)}(x,r,p,k,q) \tag{6,11}$$

7. Bilateral and bilinear generating function

THEOREM. *If we assume*

$$F[x,t]=\sum_{n=0}^{\infty}a_n M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)\,t^n,\ \ a_n\neq 0 \text{ are arbitrary} \tag{7.1}$$

constants, then

$$(1-qt)^{-\left(\frac{\alpha+k}{q}\right)}\exp\left[px^r\{1-(1-qt)^{-\frac{r}{q}}\}\right]F\left[\frac{x}{(1-qt)^{\frac{1}{q}}},\ \frac{yt}{1-qt}\right] \tag{7.2}$$

$$=\sum_{n=0}^{\infty} M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)\ b_n(y)\, t^n$$

where, $$b_n(y)=\sum_{m=0}^{n} a_m \binom{n}{m}(y)^m. \tag{7.3}$$

To prove (7.2), replace t by $tx^q y$, multiply both the sides by $x^{\alpha} e^{-px^r}$ and then operate by $e^{tT_{k,q}}$. By a simple change of variable and in view of the formulas (2.6) and (6.8), the bilateral generating relation (7.2) is established.

As an application of the above theorem we demonstrate how the well-known Hille-Hardy formula [8, *p*212]

$$(1-t)^{-\alpha-1} e^{-\left(\frac{x+y}{1-t}\right)t}\ {}_0F_1\left[-;\ \alpha+1;\ \frac{xyt}{(1-t)^2}\right]=\sum_{m=0}^{\infty}\frac{m!}{(\alpha+1)_m} L_m^{(\alpha)}(x) L_m^{(\alpha)}(y) t^m \tag{7.4}$$

and Weisner's formula [8, p. 213]

$$\sum_{m=0}^{\infty} {}_2F_1\left[-m,\ c;\ \alpha+1;\ y\right] L_m^{(\alpha)}(x) t^m=(1-t)^{c-\alpha-1}(1-t+yt)^{-c} e^{\frac{-xt}{1+t}}\cdot {}_1F_1\left[c;\ \alpha+1;\ \frac{xyt}{(1-t)(1-t+y)}\right] \tag{7.5}$$

can be obtained.

First assume $a_n=\dfrac{1}{\left(\frac{\alpha+k}{q}\right)_n}$, then from (5.8)

$$F[x,\ t]=e^{px^r+qt}\sum_{m=0}^{\infty}\frac{(-px^r)^m}{m!}\,{}_1F_1\left[-\frac{mr}{q};\ \frac{\alpha+k}{q};\ -qt\right]$$

so that (7.2) yields the interesting bilinear generating function

$$(1-qt)^{-\left(\frac{\alpha+k}{q}\right)}\exp\left[px^r-\frac{qty}{1-qt}\right]\sum_{m=0}^{\infty}\frac{\left[-p\left\{\frac{x}{(1-qt)^{\frac{1}{q}}}\right\}^r\right]^m}{m!}\,{}_1F_1\left[\frac{-mr}{q};\ \frac{\alpha+k}{q};\ \frac{qty}{1-qt}\right]$$

$$=\sum_{m=0}^{\infty}\frac{m!}{\left(\frac{\alpha+k}{q}\right)_m} M_m^{(\alpha)}(x,\ r,\ p,\ k,\ q)\ L_m^{\left(\frac{\alpha+k-q}{q}\right)}(y)\ t^m \tag{7.6}$$

This can be considered as a generalization of the Hille-Hardy formula mentioned above and indeed reduces to it when $p=q=r=1$, $k=0$ and $\alpha=\alpha+1$.

On the other hand, if we taken

$$\alpha_n=\frac{\left(\frac{c}{q}\right)_n}{\left(\frac{\alpha+k}{q}\right)_n}$$

then by (5.11)

$$F[x,t]=e^{px^r}(1-qt)^{-\frac{c}{q}}\sum_{m=0}^{\infty}\frac{(-px^r)^m}{m!}\ {}_2F_1\left[-\frac{mr}{q},\ \frac{c}{q};\ \frac{\alpha+k}{q};\ \frac{qt}{qt-1}\right].$$

In this case (7.2) becomes

$$(1-qt)^{\frac{c}{q}-\left(\frac{\alpha+k}{q}\right)}(1-qt+qty)^{-\frac{c}{q}}e^{px^r}\sum_{m=0}^{\infty}\frac{\left[-p\left\{\frac{x}{(1-qt)^{\frac{1}{q}}}\right\}^r\right]}{m}$$

$$\cdot{}_2F_1\left[-\frac{mr}{q},\ \frac{c}{q};\ \frac{\alpha+k}{q};\ \frac{qyt}{1-qt+qty}\right]$$

$$=\sum_{m=0}^{\infty}{}_2F_1\left[-m,\ \frac{c}{q};\ \frac{\alpha+k}{q};\ y\right]M_m^{(\alpha)}(x,r,p,k,q)\,t^m. \qquad (7.7)$$

Replacing α by $\alpha+1$ and substituting $p=q=r=1$, $k=0$, and after a little simplification one obtains the Weisner's formula.

While concluding we remark that several of our results will reduce to those of Srivastava and Singhal [10] when $k=0$; to those of Chatterjea [3] for $k=0$, $q=1$ and to those of Al-Salam [1] for $p=q=r=k=1$.

Department of Mathematics and Defence Laboratory
University of Roorkee Jodhpur
Roorkee, India India

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