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CERTAIN RESULTS FOR MULTIVARIATE H-FUNCTION INVOLVING JACOBI POLYNOMIALS

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1. Introduction, definition and notations

Recently, Mishra [6], Srivastava and Panda [9], Prasad and Singh [7] and several others have established certain integrals and Fourier-Jacobi expansions involving a number of special functions for one or more variables. In an attempt to unify and extend these, and other results given in the literature from time to time, we here, first evaluate some finite integrals involving the product of multivariate H-function and Jacobi polynomials with general arguments. Two interesting and useful Lemmas are then established, which can be applied to find the expansion of any given arbitrary analytic function of several complex variables in a (single or multiple) series of orthogonal functions. For the sake of illustration, we have obtained three expansion theorems for multivariate H-function in a series of Jacobi polynomials by employing our integrals and the Lemmas. Finally, their relevant connections with a number of known results are indicated briefly.

The H-function of several complex variables (or multivariate H-function) has been introduced and studied earlier by Srivastava and Panda [9] and Saxena [8]. However, in line with notation for the H-function of two variables by Goyal ([4], p.19, Eq. (1.1)) and Garg ([3], p.31, Eq. (1.1)), we shall employ the following notation, which seems to be more compact and self explainatory:

$$H[x_1, \cdots, x_r] = H\begin{matrix} o,\, n : \{m_i,\, n_i\} \\ p,\, q : \{p_i,\, q_i\} \end{matrix} \left[\begin{matrix} x_1 \\ \vdots \\ x_r \end{matrix} \middle| \begin{matrix} (a_j\,;\alpha'_j,\ \cdots,\ \alpha_j^{(r)})_{1,p} : \{(c_j^{(i)}, \varepsilon_j^{(i)})_{1,p_i}\} \\ (b_j\,;\beta'_j,\ \cdots,\ \beta_j^{(r)})_{1,q} : \{(d_j^{(i)}, \delta_j^{(i)})_{1,q_i}\} \end{matrix}\right]$$

$$= H\begin{matrix} 0,\ n : m_1,\ n_1\,; \cdots ; m_r, n_r \\ p,\ q : p_1,\ q_1\,; \cdots ; p_r,\ q_r \end{matrix} \left[\begin{matrix} x_1 \\ \vdots \\ x_r \end{matrix} \middle| \begin{matrix} (a_j\,;\alpha'_j, \cdots,\ \alpha_j^{(r)})_{1,p} : (c'_j,\ \varepsilon'_j)_{1,p_1}; \cdots ; (c_j^{(r)},\ \varepsilon_j^{(r)})_{1,p_r} \\ (b_j\,;\beta'_j, \cdots,\ \beta_j^{(r)})_{1,q} : (d'_j,\ \delta'_j)_{1,q_1}; \cdots ; (d_j^{(r)},\ \delta_j^{(r)})_{1,q_r} \end{matrix}\right]$$

$$=(1/2\pi w)^r\int_{L_1}\cdots\int_{L_r}\phi(s_1,\ \cdots,\ s_r)\prod_{i=1}^{r}\left\{\theta_i(s_i)\ (x_i)^{s_i}ds_i\right\}, \tag{1.1}$$

$$w=\sqrt{-1},\ i=1,\ \cdots,\ r$$

$$\text{where } \phi(s_1,\ \cdots,\ s_r)=\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)$$

$$\cdot\left[\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)\right]^{-1} \tag{1.2}$$

$$\theta_i(s_i)=\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma(1-c_j^{(i)}+\varepsilon_j^{(i)}s_i)$$

$$\cdot\left[\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\varepsilon_j^{(i)}s_i)\right]^{-1} \tag{1.3}$$

i in the superscript (i) stands for the number of dashes, e.g. $b^{(1)}=b'$, $b^{(2)}=b''$ and so on. The symbol $(a_j\,;\alpha'_j,\ \cdots,\ \alpha_j^{(r)})_{1,p}$ would abbreviate p-parameters $(a_1\,;\alpha'_1,\ \cdots,\ \alpha_1^{(r)}),\ \cdots,\ (a_p\,;\alpha'_p,\ \cdots,\ \alpha_p^{(r)})$ and $(c_j,\ \varepsilon_j)_{1,p}$ p-parameters $(c_1,\ \varepsilon_1),\ \cdots,\ (c_p,\ \varepsilon_p)$.

Further $n,\ p,\ q,\ m_i,\ n_i,\ p_i,\ q_i$ are non-negative integers, satisfying the inequalities $0\leq n\leq p,\ q\geq 0,\ 0\leq n_i\leq p_i,\ \ 1\leq m_i\leq q_i\ (i=1,\ \cdots,\ r)$; $a_j,\ b_j,\ c_j^{(i)},\ d_j^{(i)}$ are all complex numbers and Greek letters $\alpha_j^{(i)},\ \beta_j^{(i)},\ \varepsilon_j^{(i)},\ \delta_j^{(i)}$ are all assumed to be positive numbers for standardization purposes; the definition of the multivariate H-function given by (1.1), will however have meaning even if some of these quantities are zero.

The sequence of parameters in the integrand of (1.1) are such that none of the poles coincide, that is the poles of the integrand of (1.1) are simple. In case some of the poles coincide, then by following the method of Frobenius, the integrand in (1.1) can be evaluated in terms Psi-functions and generalized Zeta functions. In this connection the reader is referred to the work of Mathai and Saxena [5]:

The multiple integral (1.1) converges absolutely, if

$$U_i>0 \text{ and} |\arg\ x_i|<\frac{1}{2}U_i\pi, \tag{1.4}$$

$$\text{where, } U_i=-\sum_{j=n+1}^{p}\alpha_j^{(i)}-\sum_{j=1}^{q}\beta_j^{(i)}+\sum_{j=1}^{n_i}\varepsilon_j^{(i)}-\sum_{j=n_i+1}^{p_i}\varepsilon_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}\delta_j^{(i)}$$

$$(i=1,\ \cdots,\ r) \tag{1.5}$$

The conditions mentioned above are only slight varient of the conditions given by Srivastava and Panda ([9], p. 130) and Saxena ([8], p. 222). These conditions

(1.4) and (1,5) have been preferred over the conditions given by them, on account of detailed explanation given by Bushman [1, pp1—2].

Also, we have ([9], p. 131, Eq. (1.9)):

$$\left.\begin{aligned} H[x_1, \cdots, x_r] &= 0\,(|x_1|^{v_1}\cdots|x_r|^{v_r}),\ \max\{|x_1|, \cdots, |x_r|\}\to 0 \\ &= 0(|x_1|^{-w_1}\cdots|x_r|^{-w_r}),\ n=0 \text{ and } \min\{|x_1|, \cdots, |x_r|\}\to\infty \end{aligned}\right\} \quad (1,6)$$

where

$$v_i = \min_{1\le j\le m_i}\left[\mathrm{Re}\left\{d_j^{(i)}/\delta_j^{(i)}\right\}\right],\ w_i = \min_{1\le j\le n_i}\left[\mathrm{Re}\left\{(1-c_j^{(i)})/\varepsilon_j^{(i)}\right\}\right]\ (i=1, \cdots, r) \quad (1.7)$$

2. Finite integrals

In this paper the following five finite integrals (two single and three multiple) involving multivariate H-function and Jacobi and Jacobi polynomials have been evaluated:

(a) Single finite integrals

First Integral:

$$\int_0^t (t-x)^\rho x^\sigma P_u^{(\alpha,\beta)}(1-zx)\ H\left[y_1(t-x)^{k_1}x^{h_1}, \cdots, y_r(t-x)^{k_r}x^{h_r}\right]dx$$

$$= t^{\rho+\sigma+1}\sum_{w=0}^{u}\Gamma(\alpha+u+1)(-u)_w\ (\alpha+\beta+u+1)_w\left(\frac{1}{2}zt\right)^w$$

$$\cdot\ [u!w!\Gamma(\alpha+w+1)]^{-1}G_{\rho,\sigma,w}[y_1t^{h_1+k_1}, \cdots, y_rt^{h_r+k_r}], \quad (2.1)$$

where $P_u^{(\alpha,\beta)}(x)$ is the well-known Jacobi polynomials and for convenience,

$$G_{\rho,\sigma,w}[y_1, \cdots, y_r] = H^{0,\ n+2:\ \{m_i,\ n_i\}}_{p+2,\ q+1:\ \{p_i, q_i\}}\left[\begin{matrix} y_1 \\ \vdots \\ y_r \end{matrix}\left|\begin{matrix} (-\rho: k_1, \cdots, k_r), \\ (-1-\sigma-\rho-w: h_1+k_1, \cdots, h_r+k_r), \end{matrix}\right.\right.$$

$$\left.\begin{matrix} (-\sigma-w: h_1, \cdots, h_r),\ (a_j; \alpha'_j, \cdots, \alpha_j^{(r)})_{1,p}: \{(c_j^{(i)}, \varepsilon_j^{(i)})_{1,p_i}\} \\ (b_j; \beta'_j, \cdots, \beta_j^{(r)})_{1,q}: \{(d_j^{(i)}, \delta_j^{(i)})_{1,q_i}\} \end{matrix}\right] \quad (2.2)$$

The integral (2.1) converges under the following (sufficient) conditions:

(i) $\mathrm{Re}(\sigma+h_iv_i+1)>0$, $\mathrm{Re}(\rho+k_iv_i+1)>0$, $(i=1, \cdots, r)$

(ii) $h_i>0$, $k_i>0$, $U_i>0$ and $|\arg y_i|<\frac{1}{2}U_i\pi$, $(i=1, \cdots, r)$

where, U_i and v_i are defined by (1.5) and (1.7) respectively.

Second Integral:

$$\int_0^t (t-x)^\rho x^\sigma \left(1-\frac{1}{2}zx\right)^\beta p_u^{(\alpha,\beta)}(1-zx)\ H[y_1(t-x)^{k_1}x^{h_1},\ \cdots,\ y_r(t-x)^{k_r}x^{h_r}]\,dx$$

$$=t^{\rho+\sigma+1}\sum_{w=0}^{\infty}(\alpha+w+1)_u\ (-\beta-u)_w\left(\frac{1}{2}zt\right)^w\ (u!w!)^{-1}$$

$$\cdot G_{\rho,\sigma,w}[y_1 t^{h_1+k_1},\ \cdots,\ y_r t^{h_r+k_r}], \tag{2.3}$$

provided the conditions mentioned with the integral (2.1) are satisfied and the series occurring on the right of (2.3) is absolutely convergent.

(b) **Multiple finite integrals**

Third Integral:

$$\int_0^{t_1}\cdots\int_0^{t_r}\prod_{i=1}^{r}\{(t_i-x_i)^{\rho_i}x_i^{\sigma_i}\ p_{u_i}^{(\mu_i,\nu_i)}\ (1-z_ix_i)\}$$

$$\cdot H[y_1(t_1-x_1)^{k_1}x_1^{h_1},\ \cdots,\ y_r(t_r-x_r)^{k_r}x_r^{h_r}]dx_1\ \cdots dx_r$$

$$=\prod_{i=1}^{r}\{t_i^{\rho_i+\sigma_i+1}\sum_{N_i=0}^{u_i}\Gamma(\mu_i+u_i+1)(-u_i)_{N_i}(\mu_i+\nu_i+u_i+1)_{N_i}\left(\frac{1}{2}z_it_i\right)^{N_i}$$

$$\cdot[u_i!N_i!\Gamma(\mu_i+N_i+1)]^{-1}K_{\rho_i,\sigma_i,N_i}[y_1t_1^{h_1+k_1},\ \cdots,\ y_rt_r^{h_r+k_r}], \tag{2.4}$$

where

$$K_{\rho_i,\sigma_i,N_i}[y_1,\ \cdots,\ y_r]=H\begin{matrix}0,n:\{m_i,n_i+2\}\\ p,q:\{p_i+2,\ q_i+1\}\end{matrix}\left[\begin{matrix}y_1\\ \vdots\\ y_r\end{matrix}\right|$$

$$\left.\begin{matrix}(a_j;\alpha'_j,\ \cdots,\ \alpha_j^{(r)})_{1,p}:\{-(\rho_i,\ k_i),\ (-\sigma_i-N_i,h_i),\ (c_j^{(i)},\ \varepsilon_j^{(i)})_{1,p_i}\}\\ (b_j;\beta'_j,\ \cdots,\ \beta_j^{(r)})_{1,q}:\{(d_j^{(i)},\ \delta_j^{(i)})_{1,q_i},\ (-1-\rho_i-\sigma_i-N_i,h_i+k_i)\}\end{matrix}\right] \tag{2.5}$$

where, the symbol $\{(-\rho_i,\ k_i),\ (-\sigma_i-N_i,\ h_i),\ (c_j^{(i)},\ \varepsilon_j^{(i)})_{1,p_i}\}$ stands for $(-\rho_1,\ k_1),(-\sigma_1-N_1,h_1),\ (c'_j,\ \varepsilon'_j)_{1,p_1};\cdots;(-\rho_r,\ k_r),\ (-\sigma_r-N_r,\ h_r),\ (c_j^{(r)},\ \varepsilon_j^{(r)})_{1,p_r}$ and so on.

The integral (2.4) converges, under the following (sufficient) conditions:

(i) $\mathrm{Re}(\rho_i+k_iv_i+1)>0$ and $\mathrm{Re}(\sigma_i+h_iv_i+1)>0$ $(i=1,\ \cdots,\ r)$

(ii) $h_i>0,\ k_i>0,\ U_i>0$ and $|\arg\ y_i|<\frac{1}{2}U_i\pi,(i=1,\ \cdots,\ r)$

where, U_i and v_i are defined by (1.5) and (1.7) respectively.

Fourth Integral:

$$\int_0^{t_1}\cdots\int_0^{t_r}\prod_{i=1}^{r}\{(t_i-x_i)^{\rho'}x_i^{\sigma_i}\left(1-\frac{1}{2}z_ix_i\right)^{\nu_i}p_{u_i}^{(\mu_i,\nu_i)}\ (1-z_ix_i)\}$$

$$\cdot H[y_1(t_1-x_1)^{k_1}x_1^{h_1}, \cdots, y_r(t_r-x_r)^{k_r}x_r^{h_r}]\ dx_1\cdots dx_r$$

$$=\prod_{i=1}^{r}\{t_i^{\rho_i+\sigma_i+1}\sum_{N_i=0}^{\infty}(\mu_i+N_i+1)_{u_i}(-\nu_i-u_i)_{N_i}\left(\frac{1}{2}z_it_i\right)^{N_i}(u_i!N_i!)^{-1}\}$$

$$\cdot K_{\rho_i,\sigma_i,N_i}[y_1t_1^{h_1+k_1}, \cdots, y_rt_r^{h_r+k_r}], \tag{2.6}$$

provided the series occurring on the right of (2.6) is absolutely convergent and conditions given with (2.4) are satisfied.

Fifth Integral :

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{r}\{(1-x_i)^{\nu_i}x_i^{k_i}P_{u_i}^{(\mu_i,\nu_i)}(1-2x_i)P_{g_i}^{(\rho_i,\sigma_i)}(1-2x_i)\}$$

$$\cdot H[y_1x_1^{h_1}, \cdots, y_rx_r^{h_r}]\ dx_1\cdots dx_r$$

$$=\prod_{i=1}^{r}\{\sum_{N_i=0}^{g_i}\Gamma(\nu_i+u_i+1)\Gamma(\rho_i+g_i+1)(-g_i)_{N_i}(\rho_i+\sigma_i+g_i+1)_{N_i}(-1)^{u_i}$$

$$[u_i!g_i!N_i!\Gamma(\rho_i+1+N_i)]^{-1}\}L_{k_i,\nu_i,N_i}[y_1, \cdots, y_r], \tag{2.7}$$

where

$$L_{k_i,\nu_i,N_i}[y_1, \cdots, y_r]=H_{p,q\,:\,\{p_i+2,\,q_i+2\}}^{0,n\,:\,\{m_i,\,n_i+2\}}\left[\begin{matrix}y_1\\ \vdots\\ y_r\end{matrix}\middle|\begin{matrix}(a_j;\alpha'_j, \cdots, \alpha_j^{(r)})_{1,p}:\\ (b_j;\beta'_j, \cdots, \beta_j^{(r)})_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}\{(-k_i-N_i, h_i), (\mu_i-k_i-N_i, h_i), (c_j^{(i)}, \varepsilon_j^{(i)})_{1,p_i}\}\\ \{(d_j^{(i)}, \delta^{(i)})_{1,q_i}, (-\nu_i-k_i-u_i-N_i-1, h_i), (\mu_i-k_i+u_i-N_i, h_i)\}\end{matrix}\right] \tag{2.8}$$

$$(i=1, \cdots, r)$$

provided that, $\text{Re}(k_i+h_iv_i+1)>0$, $\text{Re}(\nu_i)>-1$, $|\arg y_i|<\frac{1}{2}U_i\pi$, $U_i>0$; where U_i and v_i are defined by (1.5) and (1.7) respectively.

Evaluation of formulae (2.1) and (2.3)

To derive integral (2.1), we, first write H-function occurring in the integrand in terms of Mellin-Barnes contour integral with the help of (1.1) and change the order of integration, which is justified as the series involved is finite. We then, apply a known result ([2], p. 192, Eq. (46)) in order to evaluate the x-integral and interprete the resulting contour integral as multivariate H-function with the help of (1.1) ; the integral formula follows at once.

The second integral (2.3) can be proved in a similar manner as indicated above. Here, we make use of the result ([2], p. 192, Eq. (48)) instead of (46) given in the same book and used for deriving the first integral.

Evaluation of the integral formulae (2.4), (2.6) and (2.7)

Our derivation of integral formula (2.4) makes use of the following integral:

$$\int_{0}^{t_1}\cdots\int_{0}^{t_r}\prod_{i=1}^{r}\{(t_i-x_i)^{\rho_i}x_i^{\sigma_i}P_{u_i}^{(\mu_i,\nu_i)}(1-z_ix_i)\,dx_i\}$$

$$=\prod_{i=1}^{r}\{\Gamma(\sigma_i+1)\Gamma(\mu_i+u_i+1)\Gamma(\rho_i+1)t^{\rho_i+\sigma_i+1}[u_i!\Gamma(\mu_i+1)\Gamma(\rho_i+\sigma_i+2)]^{-1}$$

$$\cdot\,{}_3F_2(-u_i,\ u_i+\mu_i+\nu_i+1,\ \sigma_i+1\,;\ \mu_i+1,\ \rho_i+\sigma_i+2\,;\ \tfrac{1}{2}z_it_i)\},\tag{2.9}$$

which holds for $\min_{1\leq i\leq r}[\mathrm{Re}(\rho_i,\ \sigma_i,\ \mu_i,\ \nu_i)]>-1$

The above formula follows readily from the known integral ([2], p. 192, Eq. (46)).

To establish (2.4), we first replace the multivariate H-function in the integrand of (2.4) by its Mellin-Barnes contour integral given by (1.1) and change the order of integration and summation, which is permissible under the conditions stated with (2.4). We then, evaluate inner most multiple x_i-integral by applying (2.9) and interpret the resulting contour integral as an H-function with the help of (1.1). The final result (2.4), together with aforementioned conditions of its convergence, will follow from the asymptotic expansion given by (1.6).

In a similar manner by applying multiple analogous of the formulae ([2], p. 192, Eq. (48)) and ([2], p. 288, Eq. (20)) instead of (2.9), we can derive easily fourth and fifth integrals.

3. Useful lemmas

In this section, we shall establish two interesting and useful Lemmas. The Lemmas are useful in the sense that they can be applied to find the expansion of any given arbitrary analytic function of several complex variables in a (single or multiple) series of orthogonal functions. The following well-known orthogonal property will be required for obtaining our Lemmas:

Orthogonal Property

Let $\{\phi_u(x)\}_{u=0}^{\infty}$ be a sequence of function defined over the interval $[a,\ b]$ and orthogonal with respect to weight function $w(x)>0$, $\forall x\in(a,\ b)$. Then, by definition, the inner product is given by:

$$(\phi_v,\ \phi_u)=\int_a^b w(x)\phi_v(x)\phi_u(x)dx=\lambda_u\delta_{uv},\tag{3.1}$$

where, as usual, δ_{uv} is the Kronecker delta, and

$$\lambda_u = \|\phi_u(x)\|^2 \neq 0, \quad (u \geq 0), \tag{3.2}$$

The above orthogonal property, plays an important role to establish the expansion of any given (analytic) function in a series (single or multiple) of orthogonal functions. Now, we state our Lemmas.

LEMMA 1. *Let the exponents ρ, σ, k_i, $h_i (i=1, \cdots, r)$ are so chosen tha' the function*

$$f(x) = (x-a)^{\sigma}(b-x)^{\rho} F[y_1(x-a)^{h_1}(b-x)^{k_1}, \cdots, y_r(x-a)^{h_r}(b-x)^{k_r}], \tag{3.3}$$

is continuous and of bounded variation, when $a \leq x \leq b$.

Then

$$f(x) = \sum_{u=0}^{\infty} f_u(y_1, \cdots, y_r)\phi_u(x), \qquad a \leq x \leq b$$

where

$$f_u(y_1, \cdots, y_r) = (\lambda_u)^{-1} \int_a^b (x-a)^{\sigma}(b-x)^{\rho} F[y_1(x-a)^{h_1}(b-x)^{k_1}, \cdots, y_r(x-a)^{h_r}(b-x)^{k_r}]$$

$$\cdot\, w(x)\phi_u(x)\, dx, \quad (u \geq 0) \tag{3.4}$$

and the integral involved in (3.4) *converges absolutely.*

Also, if $a=0$ and $b=\infty$ in Lemma 1 *then all ρ and $k_i (i=1, \cdots, r)$ should be zero.*

LEMMA 2. *If*

$$f(x_1, \cdots, x_r) = \prod_{i=1}^{r} \{(x_i - a_i)^{\sigma_i}(b_i - x_i)^{\rho_i}\} F[y_1(x_1 - a_1)^{h_1}(b_1 - x_1)^{k_1}, \cdots, y_r(x_r - a_r)^{h_r}$$

$$(b_r - x_r)^{k_r}], \tag{3.5}$$

then, analogously to Lemma 1, *we have*

$$f(x_1, \cdots, x_r) = \prod_{i=1}^{r} \left\{ \sum_{u_i=0}^{\infty} \phi_{u_i}(x_i) \right\} f_{u_1, \cdots, u_r}(y_1, \cdots, y_r), \tag{3.6}$$

$$a_i \leq x_i \leq b_i$$

where

$$f_{u_1, \cdots, u_r}(y_1, \cdots, y_r) = \prod_{i=1}^{r} \left\{ (\lambda_{u_i})^{-1} \right\} \int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} F[y_1(x_1 - a_1)^{h_1}(b_1 - x_1)^{k_1}, \cdots, y_r(x_r - a_r)^{h_r}$$

$$(b_r - x_r)^{k_r}] \prod_{i=1}^{r} \left\{ (x_i - a_i)^{\sigma_i}(b_i - x_i)^{\rho_i} w_i(x_i)\phi_{u_i}(x_i)\, dx_i \right\} \tag{3.7}$$

$$u_i \geq 0, \quad (i=1, \cdots, r)$$

provided that each of the r-integral exist.

Certainly, if each of the interval $(a_i, b_i)(i=1, \cdots, r)$ is replaced by $(0, \infty)$, then all ρ_i, k_i $(i=1, \cdots, r)$ should be zero.

4. Expansion theorems

As remarked earlier, aforementioned Lemmas can be applied to find the expansion of any arbitrary function of several complex variables in a single or multiple series of the orthogonal functions. The important examples of orthogonal set of functions are Bessel functions, Hermite polynomials, Jacobi polynomials, Bessel polynomials, Laguerre polynomials, Trigonometric functions etc. Thus, in each case and many more, it is fairly straightfoward to find a expansion (which may be formal in certain cases) of a given function of several variables in series of the orthogonal function, considered. We, here, for the sake of illustration obtain below three expansion Theorems with the help of our Lemmas and integrals given in Section 2. Indeed, these theorems unify and extend a number of such type expansions for special functions of one or more variables established from time to time by several research workers.

THEOREM 1. *With the quantities U_i, v_i and w_i given by* (1.5) *and* (1.7) *respectively, let*

(i) $h_i, k_i, \rho, \sigma > 0$, $(i=1, \cdots, r)$, $\min\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta)\} > -1$

(ii) $U_i - k_i - h_i > 0$, $|\arg y_i| < \frac{1}{2}(U_i - k_i - h_i)\pi$.

(iii) *The set* (i) *of conditions given with first integral* (2.1) *is satisfied.*

Then

$$(1-x)^{\rho} x^{\sigma} H[y_1(1-x)^{k_1}x^{h_1}, \cdots, y_r(1-x)^{k_r}x^{h_r}]$$

$$=\sum_{u=0}^{\infty}\sum_{w=0}^{u}(-u)_w(\alpha+\beta+2u+1)\Gamma(\alpha+\beta+u+w+1)[w!\Gamma(\alpha+w+1)\Gamma(\beta+u+1)]^{-1}$$

$$\cdot G_{\rho+\beta, \sigma+\alpha, w}[y_1, \cdots, y_r] P_u^{(\alpha,\beta)}(1-2x), \tag{4.1}$$

where, $u \geq 0$, $0 < x < 1$ *and the function* $G_{\rho,\sigma,w}[y_1, \cdots, y_r]$ *is defined by* (2.2).

THEOREM 2. *With U_i, v_i and w_i defined by* (1.5) *and* (1.7) *respectively, suppose that*

(i) $h_i, k_i, \rho_i, \sigma_i \geq 0$ $(i=1, \cdots, r)$ *and* $\min_{1\leq j\leq r}\{\mathrm{Re}(\mu_i), \mathrm{Re}(\nu_i)\} > -1$.

(ii) $U_i > 0$, $|\arg y_i| < \frac{1}{2}U_i\pi$ $\quad (i=1, \cdots, r)$

(iii) *The set* (i) *of conditions given with third integral holds.*

Then

$$\prod_{i=1}^{r}\left\{(1-x_i)^{\rho_i}x_i^{\sigma_i}\right\}\ H[y_1(1-x_1)^{k_1}x_1^{h_1},\ \cdots,\ y_r(1-x_r)^{k_r}x_r^{h_r}]$$

$$=\prod_{i=1}^{r}\left\{\sum_{u_i=0}^{\infty}\sum_{N_i=0}^{u_i}(-u_i)_{N_i}(\mu_i+\nu_i+2u_i+1)\Gamma(\mu_i+\nu_i+u_i+N_i+1)\,[\Gamma(\mu_i+N_i+1)\right.$$

$$\left.\Gamma(\nu_i+u_i+1)N_i!]^{-1}\,p_{u_i}^{(\mu_i,\nu_i)}(1-2x_i)\right\}K_{\rho_i+\nu_i,\sigma_i+\mu_i,N_i}\ [y_1,\ \cdots,\ y_r], \tag{4.2}$$

where, $U_i\geq 0,\ 0<x_i<1\ (i=1,\ \cdots,\ r)$ *and the function*

$K_{\rho_i,\sigma_i,N_i}\ [y_1,\ \cdots,\ y_r]$ *is given by* (2.5).

THEOREM 3. *With quantities* U_i, v_i *and* w_i *given by* (1.5) *and* (1.7) *respectively, let*

(i) $h_i,\ k_i,\ \nu_i>0,\ (i=1,\ \cdots,\ r)$ *and* $\min_{1\leq j\leq r}\{\mathrm{Re}(\rho_i),\ \mathrm{Re}(\sigma_i)\}>-1$

(ii) $U_i>0,\ |\arg y_i|<\frac{1}{2}U_i\pi\ (i=1,\ \cdots,\ r)$

(iii) *The set* (i) *of conditions given with fifth integral is satisfied.*

Then

$$\prod_{i=1}^{r}\left\{(1-x_i)^{\nu_i}x_i^{k_i}\,p_{u_i}^{(\mu_i,\nu_i)}(1-2x_i)\right\}H[y_1x_1^{h_1},\ \cdots,\ y_rx_r^{h_r}]$$

$$=\prod_{i=1}^{r}\left\{\sum_{g_i=0}^{\infty}\sum_{N_i=0}^{g_i}\Gamma(\nu_i+u_i+1)(-g_i)_{N_i}\Gamma(\rho_i+\sigma_i+g_i+N_i+1)(\rho_i+\sigma_i+2g_i+1)\right.$$

$$\left.\cdot(-1)^{u_i}[u_i!\ N_i!\ \Gamma(\rho_i+N_i+1)\Gamma(\sigma_i+g_i+1)]^{-1}\,p_{g_i}^{(\rho_i,\sigma_i)}(1-2x_i)\right\}$$

$$\cdot L_{k_i+\rho_i,\nu_i+\sigma_i,N_i}\ [y_1,\ \cdots,\ y_r], \tag{4.3}$$

where

$g_i\geq 0,\ 0<x_i<1$ *and the function* $L_{k_i,\nu_i,N_i}\ [y_1,\ \cdots,\ y_r]$ *is given by* (2.8).

PROOF OF THEOREM 1. If we put $a=0,\ b=1$ and let

$$F[y_1,\ \cdots,\ y_r]=x^{\sigma}(1-x)^{\rho}\ H[y_1(1-x)^{k_1}x^{h_1},\ \cdots,\ y_r(1-x)^{k_r}x^{h_r}]$$

in Lemma 1. Also, take classical Jacobi polynomials $\{p_u^{(\alpha,\beta)}(1-2x)\}_{u=0}^{\infty}$ as orthogonal set of functions therein.

It is well known that for Jacobi polynomials,

$w(x)=(x)^{\alpha}(1-x)^{\beta},\ a=0,\ b=1$ and

$$\lambda_u=\Gamma(\alpha+u+1)\Gamma(\beta+u+1)\,[(\alpha+\beta+2u+1)\Gamma(\alpha+\beta+u+1)u!]^{-1} \tag{4.4}$$

$$\min\{\mathrm{Re}(\alpha),\ \mathrm{Re}(\beta)\}>-1,\ u\geq 0$$

Thus, we find that

$$x^{\sigma}(1-x)^{\rho}\ H[y_1x^{h_1}(1-x)^{k_1},\ \cdots,\ y_rx^{h_r}(1-x)^{k_r}]$$

$$=\sum_{u=0}^{\infty} f_u(y_1, \cdots, y_r)\ p_u^{(\alpha,\beta)}(1-2x), \quad 0\le x\le 1 \tag{4.5}$$

where

$$f_u(y_1, \cdots, y_r)=(\lambda_u)^{-1}\int_0^1 x^{\rho+\alpha}(1-x)^{\rho+\beta} H[y_1x^{h_1}(1-x)^{k_1}, \cdots, y_rx^{h_r}(1-x)^{k_r}] \cdot p_u^{(\alpha,\beta)}(1-2x)\ dx. \tag{4.6}$$

Evaluating the integral (4.6) with the help of (2.1) (with $t=1$, $z=2$) and putting the value of $f_u(y_1, \cdots, y_r)$ so obtained and λ_u (from (4.4)) in (4.6), we arrive easily at (4.1).

PROOFS OF THEOREMS 2 AND 3.

The expansion Theorems 2 and 3 can be developed on the lines similar to Theorem 1, except that here we use the Lemma 2 and integral formulae (2.4) and (2.7) respectively instead of Lemma 1 and integral (2.1).

5. Special cases

At the out set, we should remark that our integral formulae and expansion Theorems are quite general in character. Indeed, these results can suitably be specialized to a number of known or new integrals and expansion formulas involving a large spectrum of special functions (or product of such functions). However, we mention here only some known special cases of our results.

For example, if we set $t=1$, $z=2$ and $r=2$ in (2.1), we shall fairly easily get the result by Prasad and Singh ([7], p.126, Eq.(2.1)). Again, if we put $t=1$, $z=2$, $\alpha=\beta$, $k_i=0$ $(i=1, \cdots, r)$ in (2.1), it reduces to yet another known integral due to Srivastava and Panda ([9], p.131, Eq.(2.2)). The integral by Srivastava and Panda contains a many more known integrals given by several authors as its particular cases.

On the other hand, our integral formula (2.4) would at once reduces to the integral (2.1) given by Mishra ([6], p.173) if we put $r=1$ in it. Again if $r=1$ in (2.7), we get an integral involving the product of Fox's H-function and Jacobi polynomials with general arguments. We however do not mention it explicitly.

Similarly, it can be shown easily that the Fourier Jacobi expansions obtained by Prasad and Singh ([7], p.130, Eq.(3.5), (3.8)) Srivastava and Panda ([9], p.132, Eq.(2.3)), Mishra ([6] p.176, Eq.(3.4)) and many others are special or confluent cases of our expansion Theorems 1 through 3. Further, appealing to the familiar relationships (c.f; e.g., Srivastava and Panda ([9], p.135—136)) between the Jacobi and other classes of the well-known polynomials, we can easily obtain the expansion of the H-function of several complex variables in a series (single or multiple) of other classes of orthogonal polynomials. However, we omit the details.

We conclude by remarking that a number of interesting variations of our integral formulae and expansion Theorems can be obtained when one or more k_i, h_i $(i=1, \cdots, r)$ tend to zero. The details are reasonably straightforward, and we may very well leave them as an excercise to the interested reader.

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