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ON EXISTENCE OF SOLUTIONS OF NON-LINEAR AUTONOMOUS THIRD ORDER DIFFERENTIAL EQUATION

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Abstract

In this paper a general third order differential equation encountered in the flow of fluids in general and hopefully elsewhere. Sufficient conditions for existence & uniqueness of its solutions are given.

The general third order differential euqation

$$\dddot{x}+f(x,\ \dot{x},\ \ddot{x})=0,\ \cdot=\frac{d}{dt} \tag{1}$$

such that

$$x(0)=a_1,\ \ \dot{x}(0)=a_2,\ \ \dot{x}(\infty)=a_3, \tag{2}$$

where $a_1 \geqslant 0$, $a_3>a_2>0$, $a_i \in R$, $i=1,2$.

Equations similar to (1) have been subjected to several investigations, see for example recently [1], [6], [8], [9], [10] are only a sample. Equation (1) is equivalent to the autonomous system

$$\begin{aligned} &\dot{x}=f(x),\\ \text{where}\quad &x=\mathrm{col}[x_1,\ x_2,\ x_3],\\ &f(x)=\mathrm{col}[x_2,\ x_3,\ -f(x_1,\ x_2,\ x_3)]. \end{aligned} \tag{3}$$

Our main assumptions on f are:

(i) $f(x,\ \dot{x},\ \ddot{x})>0$ for $0<a\leq\dot{x}<a_3$.

(ii) $f\in C^1$-class & monotonic decreasing for $0<a\leq\dot{x}<a_3$ and satisfies Lipschitz condition.

(iii) $f(x_1,\ a_3,\ x_3)=0$.

We define the following regions:

$$\begin{aligned} &D_1:(x_1,\ x_2,\ x_3):x_1>0,\ 0<a<x_2<a_3,\ x_3>0,\\ &D_2:(x_1,\ x_2,\ x_3):x_1>0,\ \ x_2=a>0,\ \ x_3>0,\\ &D_3:(x_1,\ x_2,\ x_3):x_1>0,\ 0<a<x_2<a_3,\ x_3=0. \end{aligned} \tag{4}$$

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The relation between a, a_2, a_3 & b are given by $a<a_2<a_3$, $a_1\geq 0$, $b\geq 0$.

REMARK. It is clear that

$$P(t)=\text{col}[a_3t+a_1,\ a_3,\ 0] \tag{5}$$

is a solution of the system (5). We shall depend on the uniqueness of solutions to be sure that no solution of (3) cuts the solution (5).

THEOREM 1. *Under the previous assumptions, equation* (1) *with condition* (2) *has a unique solution.*

PROOF. First we assume that a solution of (1) which satisfying $x(0)=a_1$, $\dot{x}(0)=a_2$ stays in D_1 & thus if all conditions (2) are satisfied, then the solution of (1) exists if $b>0$ otherwise the solution would go out D_1 via D_3. From (3) we notice that:

(i) $\dot{x}_2=x_3>0$, thus x_2 is an increasing function, but $x_3=\dot{x}_2$ is a decreasing function since $\dot{x}_3<0$. Since the solution is assumed to stay in D_1, then x_2 is bounded & hence $x_3\longrightarrow 0$ i.e. $\ddot{x}\longrightarrow 0$.

(ii) x_3 is clearly bounded between 0 and b.

(iii) $\dot{x}_1=x_2$ implies $x_1\longrightarrow\infty$.

(iv) $\dot{x}(\infty)=a_3$, i.e. $x_2\longrightarrow a_3$.

We shall prove that some solutions corresponding to (a_1, a_2, b), a_1 & a_2 are fixed, will leave D_1 via D_2 & some will leave via D_3. Since the required solutions as a function of b, vary continuously and no solution leaves via the boundary (5). Thus we can conclude that some (unique) solution stays in D_1, the required case. For given fixed numbers a_1 and a_2, as for D_3, there exists values of b the solutions leave via D_3. This takes place in case $b=0$ thus by continuity this for which case takes place for small values of $b>0$.

It is clear that for sufficiently large b, the solution will leave D_1 via D_2. Thus we proved that if all conditions (2) are satisfied then there exist solutions.

Secondly we shall show that the solutions stay in D_1 are unique. On the contrary we suppose that there are two solutions of (1) which did not leave D_1 via D_2 or via D_3, i.e. the two solutions stay in the region from which

$$x>0,\ 0<a<\dot{x}<a_3,\ \ddot{x}>0.$$

Using the substitution:

$$\dot{x}=y$$

then the second and third derivatives of x have the forms:

$$\ddot{x}=\dot{x}\,\frac{dy}{dx}=y\,\frac{dy}{dx}=y\,y', \qquad '=\frac{d}{dx}$$

$$\dddot{x}=y\left(\frac{dy}{dx}\right)^2+y^2\frac{d^2y}{dx^2}=yy'+y^2\,y''.$$

Thus equation (1) is reduced to

$$y''=-\frac{1}{y^2}\,f(x,\,y,\,y')-\frac{1}{y}\,(y')^2=9(x,\,y,\,y'). \tag{6}$$

Now if there are two solutions for (1), then consequently equation (6) has two solutions, say, $y(x)$ and $z(x)$, which satisfying the boundary conditions, i. e., $y(a_1)=z(a_1)=a_2$, $y(\infty)=z(\infty)=a_3$. (7)

Let $y(x)\neq z(x)$ for some $x>a_1$. It is clear, from (7), that the two solutions are identical at $x=a_1$ and having the same limit a_3 as $x\longrightarrow\infty$. Thus we have one of the following:

(i) The two solutions are equal on the whole interval, hence the solution of (1) is unique.

or;

(ii) at some point $x>a_1$, one solution $z(x)$, say, lies above $y(x)$. Thus the function $u(x)=z(x)-y(x)$ has positive maxima on the interval at some point c. Thus at the maximum c of $u(x)$ we have:

(i) $z(c)>y(c)$ i. e. $u(c)>0$.

(ii) $u'(c)=z'(c)-y'(c)=0$ i. e $y'(c)=z'(c)$.

(iii) $u''(c)\leqslant 0$ i. e. $y''(c)\geqslant z''(c)$. (8)

The new function $g(x,\,y',\,y'')$ defined in (6) is an increasing function in its variable y. Thus

$$u''(c)=g(c,\,z(c),\,z'(c))-g(c,\,y(c),\,y'(c))>0$$

which contradicts (8).

Thus the solution is unique in both cases. This completes the proof.

GENERAL REMARK. If equation (1) is considered as unperturbed equation and has a unique periodic solution which is stable see [4], we can still consider the existence and stability of the periodic solution of the three corresponding perturbed equations, where μ alway denotes a small parameter:

(i) $\dddot{u}+f(u,\,\dot{u},\,\ddot{u})=\mu g(u,\,\dot{u},\,\ddot{u},\,\mu)$,

where f & g are analytic in all variables (see [2]).

(ii) $\dddot{u}+f(u,\,\dot{u},\,\ddot{u})=\mu g(t,\,u,\,\dot{u},\,\ddot{u},\,\mu)$,

where f & g are analytic in all variable and g is periodic in t (see [7]).

(iii) $\dddot{u}=f(u, \dot{u}, \ddot{u})=\mu g\left(\frac{t}{w}, u, \dot{u}, \ddot{u}, w, \mu\right)$,

where f & g are analytic in all variables, periodic in u, and g is also periodic in t (see [3], [5]).

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