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GENERATING FUNCTIONS FOR G-FUNCTION (II)

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1. Introduction

This paper is in continuation of our previous papers [2,5], where we have obtained generating functions for G-function. In this paper we add four more formulae to our previous results. The results obtained are general in character and include as particular cases the results obtained by Meijer [4]. The importance of G-function derives largely from the possibility of expressing by means of G-symbol a great many of the special functions appearing in Applied Mathematics. For the definition of a G-function see [3, p.369 (7)]. In the investigation we use the formula [1, p.108(1)]

$$(1)\quad {}_2F_1(a,\, b;\, c;\, x)=A_1\ {}_2F_1\,(a,\, b;\, a+b-c;\, 1-x)$$
$$+A_2(1-x)^{c-a-b}\ {}_2F_1(c-a,\, c-b;\, c-a-b+1;\, 1-x),$$

where

$$A_1=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad A_2=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

and $|\arg(1-x)|<\pi$.

2. The first formula to be proved is

$$(2)\quad \sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\beta)_r r!}G^{m+1,\,n}_{p,\,q+1}\left(x\,\middle|\,\begin{matrix}a_p\\ \lambda+r,\ b_q\end{matrix}\right)t^r=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}$$

$$\sum_{r=0}^{\infty}\frac{(\alpha)_r(1-t)^r}{r!}G^{m+2,\,n+1}_{p+2,\,q+3}\left(x\,\middle|\,\begin{matrix}1+\alpha-\beta+\lambda,\ a_p,\ \alpha-\beta+\lambda+r+1\\ \lambda+r,\ \alpha-\beta+\lambda+1,\ b_q,\ 1-\beta+\lambda\end{matrix}\right)$$

$$+\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\frac{(\beta-\alpha)_r(1-t)^{\beta-\alpha-\lambda+r}}{r!}G^{m+1,\,n+2}_{p+2,\,q+3}\left(x-xt\,\middle|\,\begin{matrix}1-\beta+\lambda-r,\ \alpha-\beta+\lambda,\ a_p\\ \alpha-\beta+\lambda,\ b_q,\ 1-\beta+\lambda,\ \alpha-\beta+\lambda-r\end{matrix}\right),$$

provided that $R(\beta)>0$, $|\arg(1-t)|<\pi$, $2(m+n)>p+q-1$, $|\arg x|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q+\frac{1}{2}\right)\pi$.

PROOF. To prove (2) we substitute the contour integral for G-function [3, p. 369(7)] in the left side of (2), change the order of integration and summation, which is justified under the condition stated with (2), we have

$$(3)\quad \frac{1}{2\pi i}\int_C A\Gamma(\lambda-s)x^s\,{}_2F_1(\alpha,\ \lambda-s;\ \beta;\ t)ds,$$

where $A=\dfrac{\prod_{j=1}^{n}\Gamma(1-a_j+s)\prod_{j=1}^{m}\Gamma(b_j-s)}{\prod_{j=n+1}^{p}\Gamma(a_j-s)\prod_{j=m+1}^{q}\Gamma(1-b_j+s)}$.

Using (2) in (3) and changing the order of summation and integration, we get

$$(4)\quad \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\sum_{r=0}^{\infty}\frac{(\alpha)_r(1-t)^r}{r!}\ \frac{1}{2\pi i}\int_C A\frac{\Gamma(\beta-\alpha-\lambda+s)}{\Gamma(\beta-\lambda+s)}$$

$$\frac{\Gamma(\alpha+r-s)\Gamma(\alpha-\beta+\lambda+1-s)}{\Gamma(\alpha-\beta+\lambda+r+1-s)}x^s ds+\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\frac{(\beta-\alpha)_r}{r!}$$

$$\times(1-t)^{\beta-\alpha-\lambda+r}\frac{1}{2\pi i}\int_C\frac{A\Gamma(\alpha-\beta+\lambda-s)\Gamma(\beta-\lambda+r+s)\Gamma(\beta-\alpha-\lambda+1+s)x^s(1-t)^s}{\Gamma(\beta-\lambda+s)\Gamma(\beta-\alpha-\lambda+1+s)}ds.$$

Interpreting (4) with the help of definition of G-function [3, p. 369(7)], we get the right side of (2).

In case we take $\beta=\alpha+\omega$, $\omega=0, 1, 2, \cdots$ in (2), it reduces to the following interesting formula

$$(5)\quad \sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\alpha+\omega)_r r!}G_{p,\ q+1}^{m+1,\ n}\left(x\left|\begin{matrix}a_p\\ \lambda+r,\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\alpha+\omega)}{\Gamma(\alpha)}\sum_{r=0}^{\omega}\frac{(-\omega)_r(1-t)^{\omega-\lambda+r}}{r!}$$

$$G_{p+2,\ q+3}^{m+1,\ n+2}\left(x-xt\left|\begin{matrix}1-\alpha-\omega+\lambda-r,\ \lambda-\omega,\ a_p\\ -\omega+\lambda,\ b_q,\ 1-\alpha-\omega+\lambda,\ -\omega+\lambda-r\end{matrix}\right.\right).$$

If we put $\omega=0$, (5) gives a known result due to Meijer [4, p. 487(46)].

3. The second formula to be proved is

$$(6)\quad \sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r}{r!\,(\beta)_r}G_{p,\ q+1}^{m,\ n}\left(x\left|\begin{matrix}a_p\\ b_q,\ \lambda+r\end{matrix}\right.\right)t^r=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}(1-t)^r$$

$$G_{p+3,\ q+4}^{m+1,\ n+2}\left(x\left|\begin{matrix}1+\lambda-r,\ \beta+\lambda-\alpha,\ a_p,\ \beta+\lambda\\ \beta-\alpha+\lambda,\ b_q,\ \lambda,\ 1+\lambda,\ \beta+\lambda-\alpha-r\end{matrix}\right.\right)$$

$$+\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\frac{(\beta-\alpha)_r}{r!}(1-t)^{\beta-\alpha+\lambda+r}$$

$$G^{m+2,\ n+1}_{p+3,\ q+4}\left(\frac{x}{1-t}\left|\begin{matrix}1-\alpha+\beta+\lambda,\ a_p,\ \beta+\lambda,\ 1+\beta-\alpha+\lambda+r\\ \beta+\lambda+r,\ 1+\beta-\alpha+\lambda,\ b_q,\ \lambda,\ 1+\lambda\end{matrix}\right.\right),$$

where $R(\beta)>0$, $|\arg(1-t)|<\pi$, $2(m+n)>p+q+1$, $|\arg x|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2}\right)\pi$.

(6) can be proved in the same way as (2).

In particular we take $\beta=\alpha+\omega$, $\omega=0, 1, 2, \cdots$ in (6), it yields the following interesting result:

$$(7)\ \sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r}{(\alpha+\omega)_r r!}G^{m,\ n}_{p,\ q+1}\left(x\left|\begin{matrix}a_p\\ b_q,\ \lambda+r\end{matrix}\right.\right)t^r=\frac{\Gamma(\alpha+\omega)}{\Gamma(\alpha)}$$

$$\sum_{r=0}^{\omega}\frac{(-\omega)_r(1-t)^{\omega+\lambda+r}}{r!}G^{m+2,\ n+1}_{p+3,\ q+4}\left(\frac{x}{1-t}\left|\begin{matrix}1+\omega+\lambda,\ a_p,\ \alpha+\omega+\lambda,\ 1+\omega+\lambda+r\\ \alpha+\omega+\lambda+r,\ 1+\omega+\lambda,\ b_q,\ \lambda,\ 1+\lambda\end{matrix}\right.\right)$$

Taking $\omega=0$ in (7), we have the result due to Meijer [4, p. 486 (42)].

4. The third formula to be proved is

$$(8)\ \sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\beta)_r r!}G^{m,\ n+1}_{p+1,\ q}\left(x\left|\begin{matrix}\lambda-r,\ a_q\\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}(1-t)^r$$

$$G^{m+1,\ n+2}_{p+3,\ q+2}\left(x\left|\begin{matrix}\lambda-r,\ \lambda-\beta-\alpha-1,\ a_p,\ \beta+\lambda-1\\ \beta-\alpha+\lambda-1,\ b_q,\ \beta-\lambda-\alpha-r-1\end{matrix}\right.\right)+\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\frac{(\beta-\alpha)_r}{r!}$$

$$(1-t)^{\beta-\alpha+\lambda+r-1}G^{m+2,\ n+1}_{p+3,\ q+2}\left(\frac{x}{1-t}\left|\begin{matrix}\beta+\lambda-\alpha,\ a_p,\ \beta+\lambda-1,\ \beta-\alpha+\lambda+r\\ \beta+\lambda+r-1,\ \beta-\alpha+\lambda,\ b_q\end{matrix}\right.\right),$$

provided that $R(\beta)>0$, $|\arg(1-t)|<\pi$, $2(m+n)>p+q-1$, $|\arg x|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q+\frac{1}{2}\right)\pi$.

(8) can be proved in the same way as (2).

In particular we take $\beta=\alpha+\omega$, $\omega=0, 1, 2, \cdots$ in (8), we get the following formula

$$(9)\ \sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\alpha+\omega)_r}G^{m,\ n+1}_{p+1,\ q}\left(x\left|\begin{matrix}\lambda-r,\ a_p\\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\alpha+\omega)}{\Gamma(\alpha)}\sum_{r=0}^{\omega}\frac{(-\omega)_r}{r!}(1-t)^{\omega+\lambda+r-1}$$

$$G^{m+2,\ n+1}_{p+3,\ q+2}\left(\frac{x}{1-t}\left|\begin{matrix}\omega+\lambda,\ a_q,\ \alpha+\omega+\lambda-1,\ \omega+\lambda+r\\ \alpha+\omega+\lambda+r-1,\ \omega+\lambda,\ b_q\end{matrix}\right.\right).$$

In case $\omega=0$, we get a known formula due to Meijer [4, p. 487 (47)].

5. The fourth formula to be proved is

$$(10)\ \sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r}{(\beta)_r r!}G_{p+1,\ q}^{m,\ n}\left(x\left|\begin{matrix}a_p,\ \lambda-r\\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}(1-t)^r$$

$$G_{p+4,\ q+3}^{m+1,\ n+2}\left(x\left|\begin{matrix}\lambda-r,\ \beta+\lambda-\alpha-1,\ a_q,\ \beta+\lambda-1\\ \beta-\alpha+\lambda-1,\ b_q,\ \lambda,\ \beta+\lambda-\alpha-r-1\end{matrix}\right.\right)+\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\frac{(\beta-\alpha)_r}{r!}$$

$$(1-t)^{\beta-\alpha+\lambda+r-1}G_{p+4,\ q+3}^{m+2,\ n+1}\left(\frac{x}{1-t}\left|\begin{matrix}\beta+\lambda-\alpha,\ a_p,\ \lambda,\ \beta+\lambda-1,\ \beta-\alpha+\lambda+r\\ \beta+\lambda+r-1,\ \beta-\alpha+\lambda,\ b_q,\ \lambda\end{matrix}\right.\right),$$

provided that $R(\beta)>0$, $|\arg(1-t)|<\pi$, $2(m+n)>p+q+1$, $|\arg x|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2}\right)\pi$.

(10) can be proved in the same way as (2).

In particular if we take $\beta=\alpha+\omega$, $\omega=0, 1, 2, \cdots$ in (10), we have

$$(11)\ \sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r}{(\alpha+\omega)_r r!}G_{p+1,\ q}^{m,\ n}\left(x\left|\begin{matrix}a_p,\ \lambda-r\\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\alpha+\omega)}{\Gamma(\alpha)}\sum_{r=0}^{\omega}\frac{(-\omega)_r}{r!}(1-t)^{\omega+\lambda+r-1}$$

$$G_{p+4,\ q+3}^{m+2,\ n+1}\left(\frac{x}{1-t}\left|\begin{matrix}\omega+\lambda,\ a_q,\ \lambda,\ \alpha+\omega+\lambda-1,\ \omega+\lambda+r\\ \alpha+\omega+\lambda+r-1,\ \omega+\lambda,\ b_q,\ \lambda\end{matrix}\right.\right).$$

In case we take $\omega=0$ in (11), it yields a formula due to Meijer [4, p. 487(48)].

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