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GENERATING FUNCTIONS FOR G-FUNCTION (I)

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1. Introduction

Generating functions play a major role in the study of polynomial sets. Boas and Buck [14], Brafman [6—8], Carlitz [10, 11], Rainville [4, 5], Smith [13], Watson [9] and Weisner [12] have studied the generating functions for various classical polynomials. In 1952 Meijer [2, 3] published the generating functions for G-function. For the definition of G-function see [2, p. 359 (7)]. The importance of G-function derives largely from the possibility of expressing by means of G-symbol a great many of the special functions appearing in applied Mathematics. The object of this paper is to prove the following generating functions for G-functions

$$\sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!(\beta)_r} G_{p,q+1}^{m+1,n}\left(x \middle| \begin{matrix} a_p \\ \lambda+r, b_q \end{matrix}\right) t^r = \frac{\Gamma(\beta)(1-t)^{-\alpha}}{\Gamma(\beta-\alpha)} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} (1-t)^{-r}$$

$$G_{p+2,\,q+3}^{m+1,\,n+2}\left(x \middle| \begin{matrix} 1-\beta+\lambda-r,\ \lambda-\alpha,\ a_p \\ \lambda-\alpha,\ b_q,\ 1-\beta+\lambda,\ \lambda-\alpha-r \end{matrix}\right) + \frac{\Gamma(\beta)(1-t)^{-\lambda}}{\Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(\beta-\alpha)_r (1-t)^{-r}}{r!}$$

$$G_{p+2,\,q+3}^{m+2,\,n+1}\left(x-xt \middle| \begin{matrix} \lambda-\alpha+1,\ a_p,\ \lambda-\alpha+r+1 \\ \lambda+r,\ \lambda-\alpha+1,\ b_q,\ \lambda-\beta+1 \end{matrix}\right), \tag{1}$$

where $R(t) > \frac{1}{2}$, $2(m+n) > p+q-1$, $|\arg x| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q+\frac{1}{2}\right)\pi$.

2. Generating functions

$$\sum_{r=0}^{\infty} \frac{(-1)^r(\alpha)_r}{r!(\beta)_r} G_{p,\,q+1}^{m,\,n}\left(x \middle| \begin{matrix} a_p \\ b_q,\ \lambda+r \end{matrix}\right) t^r = \frac{\Gamma(\beta)(1-t)^{-\alpha}}{\Gamma(\beta-\alpha)} \sum_{r=0}^{\infty} \frac{(\alpha)_r (1-t)^{-r}}{r!}$$

$$G_{p+3,\,q+4}^{m+2,\,n+1}\left(x \middle| \begin{matrix} 1+\alpha+\lambda,\ a_p,\ \beta+\lambda,\ 1+\alpha+\lambda+r \\ \beta+\lambda+r,\ 1+\alpha+\lambda,\ b_q,\ \lambda,\ \lambda+1 \end{matrix}\right) + \frac{\Gamma(\beta)(1-t)^{\lambda}}{\Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(\beta-\alpha)_r (1-t)^{-r}}{r!}$$

$$G_{p+3,\,q+4}^{m+1,\,n+2}\left(\frac{x}{1-t} \middle| \begin{matrix} 1+\lambda-r,\ \alpha+\lambda,\ a_p,\ \beta+\lambda \\ \alpha+\lambda,\ b_q,\ \lambda,\ 1+\lambda,\ \alpha+\lambda-r \end{matrix}\right), \tag{2}$$

where $R(t) > \frac{1}{2}$, $2(m+n) > p+q+1$, $|\arg x| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2}\right)\pi$.

$$\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!(\beta)_r}G_{p+1,\,q}^{m,\,n+1}\left(x\left|\begin{matrix}\lambda-r,\ a_p\\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\beta)(1-t)^{-\alpha}}{\Gamma(\beta-\alpha)}\sum\frac{(\alpha)_r(1-t)^{-r}}{r!}$$

$$G_{p+3,\,q+3}^{m+2,\,n+1}\left(x\left|\begin{matrix}\alpha+\lambda,\ a_p,\ \alpha+\lambda+r,\ \beta+\lambda-1\\ \alpha+\lambda,\ \beta+\lambda+r-1,\ b_q\end{matrix}\right.\right)+\frac{\Gamma(\beta)(1-t)^{\lambda-1}}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\frac{(\beta-\alpha)_r}{r!}$$

$$(1-t)^{-r}G_{p+3,\,q+2}^{m+1,\,n+2}\left(\frac{x}{1-t}\left|\begin{matrix}\lambda-r,\ \alpha+\lambda-1,\ a_p,\ \beta+\lambda-1\\ \alpha+\lambda-1,\ b_q,\ \alpha+\lambda-r-1\end{matrix}\right.\right), \tag{3}$$

where $R(t)>\frac{1}{2}$, $2(m+n)>p+q-1$, $|\arg x|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q+\frac{1}{2}\right)\pi$.

$$\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r}{r!(\beta)_r}G_{p+1,\,q}^{m,\,n}\left(x\left|\begin{matrix}a_p,\ \lambda-r\\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\beta)(1-t)^{-\alpha}}{\Gamma(\beta-\alpha)}\sum_{r=0}^{\infty}\frac{(\alpha)_r(1-t)^{-r}}{r!}$$

$$G_{p+4,\,q+3}^{m+2,\,n+1}\left(x\left|\begin{matrix}\alpha+\lambda,\ a_p,\ \lambda,\ \alpha+\lambda+r,\ \beta+\lambda-1\\ \alpha+\lambda,\ \beta+\lambda-r-1,\ b_q,\ \lambda\end{matrix}\right.\right)+\frac{\Gamma(\beta)(1-t)^{\lambda-1}}{\Gamma(\alpha)}\sum_{r=0}^{\infty}\frac{(\beta-\alpha)_r(1-t)^{-r}}{r!}$$

$$G_{p+4,\,q+3}^{m+1,\,n+2}\left(\frac{x}{1-t}\left|\begin{matrix}\lambda-r,\ \alpha+\lambda-1,\ a_q,\ \lambda,\ \beta+\lambda-1\\ \alpha+\lambda-1,\ b_q,\ \lambda,\ \alpha+\lambda-r-1\end{matrix}\right.\right), \tag{4}$$

where $R(t)>\frac{1}{2}$, $2(m+n)>p+q+1$, $|\arg x|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2}\right)\pi$.

We shall give below the proof of (1) and (2), (3) and (4) can be proved in the same way. To prove (1), we substitute the contour integral of G-function [2, p. 369(7)] in the left side (1). Changing the order of integration and summation, which is justified under the conditions mentioned with (1), we have

$$\frac{1}{2\pi i}\int_C A\Gamma(\lambda-s)\,{}_2F_1(\alpha,\ \lambda-s,\ \beta\,;t)ds, \tag{5}$$

where C is a suitable contour and

$$A=\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}x^s.$$

On using the formula due to Erdelyi [1, p. 109 (3)] and expressing the hypergeometric series in power series, again changing the order of summation and integration and interpreting the result with the help of the formula [2, p. 369 (7)], we get the right side of (1).

We mention below some interesting particular cases of (1), (2), (3) and (4). If we take $\beta=\alpha+l$, where $l=0, 1, 2, \cdots$, in (1), it gives

$$\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!(\alpha+l)_r}G_{p,\ q+1}^{m+1,\ n}\left(x\left|\begin{matrix}a_p\\ \lambda+r,\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\alpha+l)}{\Gamma(\alpha)}$$

$$(1-t)^{-\lambda}\sum_{r=0}^{l}\frac{(-l)_r(1-t)^{-r}}{r!}G_{p+2,\ q+3}^{m+2,\ n+1}\left(x-xt\left|\begin{matrix}\lambda-\alpha+1,\ a_p,\ \lambda-\alpha+r+1\\ \lambda+r,\ \lambda-\alpha+1,\ b_q,\ \lambda-\alpha-l+1\end{matrix}\right.\right) \quad (4)$$

In case $l=0$, we get a result due to Meijer [3, p. 487(46)].

We obtain the following results from (2), (3) and (4) by taking $\beta=\alpha+l$, where $l=0,\ 1,\ 2,\ \cdots$,

$$\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r}{r!(\alpha+l)_r}G_{p,\ q+1}^{m,\ n}\left(x\left|\begin{matrix}a_p\\ b_q,\ \lambda+r\end{matrix}\right.\right)t^r=\frac{\Gamma(\alpha+l)(1-t)^{\lambda}}{\Gamma(\alpha)}\sum_{r=0}^{l}\frac{(-l)_r}{r!}$$

$$(1-t)^{-r}G_{p+3,\ q+4}^{m+1,\ n+2}\left(\frac{x}{1-t}\left|\begin{matrix}1+\lambda-r,\ \alpha+\lambda,\ a_p,\ \alpha+l+\lambda\\ \alpha+\lambda,\ b_q,\ \lambda,\ 1+\lambda,\ \alpha+\lambda-r\end{matrix}\right.\right), \quad (5)$$

$$\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!\,(\beta)_r}G_{p+1,\ q}^{m,\ n+1}\left(x\left|\begin{matrix}\lambda-r,\ a_p\\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\alpha+l)(1-t)^{\lambda-1}}{\Gamma(\alpha)}\sum_{r=0}^{l}\frac{(-l)_r(1-t)^{-r}}{r!}$$

$$G_{p+3,\ q+2}^{m+1,\ n+2}\left(\frac{x}{1-t}\left|\begin{matrix}\lambda-r,\ \alpha+\lambda-1,\ a_p,\ \alpha+l+\lambda-1\\ \alpha+\lambda-1,\ b_q,\ \alpha+\lambda-r-1\end{matrix}\right.\right) \quad (6)$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha)_r}{r!(\alpha+l)_r}G_{p+1,\ q}^{m,\ n}\left(x\left|\begin{matrix}a_p,\ \lambda-r\\ b_q\end{matrix}\right.\right)t^r=\frac{\Gamma(\alpha+l)(1-t)^{\lambda-1}}{\Gamma(\alpha)}\sum_{r=0}^{l}\frac{(-l)_r(1-t)^{-r}}{r!}$$

$$G_{p+4,\ q+3}^{m+1,\ n+2}\left(\frac{x}{1-t}\left|\begin{matrix}\lambda-r,\ \alpha+\lambda-1,\ a_p,\ \lambda,\ \alpha+l+\lambda-1\\ \alpha+\lambda-1,\ b_q,\ \lambda,\ \alpha+\lambda-r-1\end{matrix}\right.\right) \quad (7)$$

If We take $l=0$ in (5), (6) and (7), we get results due to Meijer [3, p. 486 (42), p. 487(47), (48)].

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