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ON GROUPS OF NEAT AND PURE-HIGH EXTENSIONS

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It is not difficult to verify that for a torsion group B, $\text{Next}(B, A)=0=\text{Hext}_p(B, A)$, whenever A is a divisible group or an elementary p-group. A natural question arises what will be the nature of $\text{Next}(B, A)$ and $\text{Hext}_p(B, A)$ for a torsion group B, if A changes? In this paper, we answer this question and prove that for a torsion free group A, $\text{Next}(Q/Z, A)$ is reduced algebraically compact group, while $\text{Hext}_p(Q/Z, A)=0$. Furthermore, we investigate that if A_t is the torsion part of A, $\text{Next}(Q/Z, A_t)$ will be algebraically compact, whenever $\text{Next}(Q/Z, A)$ is. The behaviour of $\text{Hext}_p(Q/Z, A_t)$ is exactly similar. We connect in this paper Next to Hom, with the remark that an analogous result is valid for Hext_p.

The exact sequence $0\longrightarrow A\longrightarrow G\longrightarrow B\longrightarrow 0$ is called *neat exact* if A is a neat subgroup of G. The elements of the group $\text{Next}(B, A)$ are the neat exact sequences. The above neat exact sequence yields for any group K the exact sequences.

$$0\longrightarrow \text{Hom}(K, A)\longrightarrow \text{Hom}(K, G)\longrightarrow \text{Hom}(K, B)\longrightarrow \text{Next}(K, A)\longrightarrow \text{Next}(K, G)\longrightarrow \text{Next}(K, B)\longrightarrow 0$$

$$0\longrightarrow \text{Hom}(B, K)\longrightarrow \text{Hom}(G, K)\longrightarrow \text{Hom}(A, K)\longrightarrow \text{Next}(B, K)\longrightarrow \text{Next}(G, K)\longrightarrow \text{Next}(A, K)\longrightarrow 0$$

The exact sequence $0\longrightarrow A\longrightarrow G\longrightarrow B\longrightarrow 0$ is a *purehigh extension* if and only if there exists a subgroup K of G such that A is maximal disjoint from K and $(A+K)/K$ is pure in G/K. The elements of the group $\text{Hext}_p(B, A)$ are the pure-high exact sequences. In general we adopt the notations used in [1].

To start with, we prove that the group of neat extensions of A by B is cotorsion.

LEMMA 1. $\text{Next}(B, A)$ *is for all groups A and B a cotorsion group.*

PROOF. For arbitrary groups A and B the factor group $\text{Ext}(B, A)/\text{Next}(B, A)$ is reduced. The exact sequence

$$0\longrightarrow \text{Next}(B, A)\longrightarrow \text{Ext}(B, A)\longrightarrow \text{Ext}(B, A)/\text{Next}(B, A)\longrightarrow 0$$

yields the exact sequence

$\mathrm{Hom}(Q, \mathrm{Ext}(B, A)/\mathrm{Next}(B, A)) \longrightarrow \mathrm{Ext}(Q, \mathrm{Next}(B, A)) \longrightarrow \mathrm{Ext}(Q, \mathrm{Ext}(B, A)$
The first group is 0 since Q is divisible and the factor group is reduced. Also $\mathrm{Ext}(B, A)$ is cotorsion for all groups A and B (see Theorem 54.6 of [1]). Hence the last group vanishes, and $\mathrm{Ext}(Q, \mathrm{Next}(B, A))=0$
Thus $\mathrm{Next}(B, A)$ is cotorsion.

A proof analogous to theorem 53.3 of [1] establishes that $\mathrm{Next}(C, A) = \bigcap_{p\in P} P\mathrm{Ext}(C, A)$ that is, $\mathrm{Next}(C, A)$ is the Frattini subgroup of $\mathrm{Ext}(C, A)$. From theorem 52.2 of [1] we deduce the following lemma.

LEMMA 2. *Let $\{G_i; i\in I\}$ be a family of groups, for any group H*

$$\mathrm{Next}(\bigoplus_{i\in I} G_i, H) \cong \prod_{i\in I} \mathrm{Next}(G_i, H)$$

$$\mathrm{Next}(H, \prod_{i\in I} G_i) \cong \prod_{i\in I} \mathrm{Next}(H, G_i)$$

PROOF. Since Frattini subgroups of two isomorphic groups are isomorphic, and Frattini subgroup of a direct product is the direct product of the Frattini subgroups the isomorphism

$$\mathrm{Ext}(\bigoplus_{i\in I} G_i, H) \cong \prod_{i\in I} \mathrm{Ext}(G_i, H)$$

$$\Longrightarrow \bigcap_{p\in P} p(\mathrm{Ext}(\bigoplus_{i\in I} G_i, H)) \cong \bigcap_{p\in P} p(\prod_{i\in I} \mathrm{Ext}(G_i, H))$$

$$\Longrightarrow \bigcap_{p\in P} p(\mathrm{Ext}(\bigoplus_{i\in I} G_i, H) \cong \prod_{i\in I} \bigcap_{p\in P} p\, \mathrm{Ext}(G_i, H)$$

$$\Longrightarrow \mathrm{Next}(\bigoplus_{i\in I} G_i, H) \cong \prod_{i\in I} \mathrm{Next}(G_i, H).$$

The proof for second isomorphism runs dually.

Now we discuss the behaviour of $\mathrm{Next}(Q/Z, A)$ and $\mathrm{Hext}_p(Q/Z, A)$, when A is a torsion free group

THEOREM 1. *Let D be the divisible hull of any torsion free group A, then for any monomorphism g of A into $D \oplus \prod_{p\in P}(A/pA)$*

$$\mathrm{Next}(Q/Z, A) \cong \mathrm{Hom}(Q/Z, (D \oplus \prod_{p\in P}(A/pA))/gA)$$

Hence $\mathrm{Next}(Q/Z, A)$ *is a reduced algebraically compact group*

PROOF. Since D is the divisible hull of A the sequence

$$0 \longrightarrow A \xrightarrow{f} D \longrightarrow D/A \longrightarrow 0$$

is exact. Define a monomorphism g of A into $D \oplus \prod_{p\in P}(A/pA)$ such that $g(a) = (f(a), \{a+pA\})$, $a\in A$. (then by lemma 4 of [2] or by [4]) the sequence

$$0 \longrightarrow A \xrightarrow{g} D \oplus \prod_{p \in P}(A/pA) \longrightarrow (D \oplus \prod_{p \in P}(A/pA))/gA) \longrightarrow 0$$

is neat exact and yields the exact sequence

$$\text{Hom}(Q/Z, D \oplus \prod_{p \in P}(A/pA)) \longrightarrow \text{Hom}(Q/Z, (D \oplus \prod_{p \in P}(A/pA))/gA)$$
$$\longrightarrow \text{Next}(Q/Z, A) \longrightarrow \text{Next}(Q/Z, (D \oplus \prod_{p \in P}(A/pA)))$$

Now, $\text{Hom}(Q/Z, D \oplus \prod_{p \in P}(A/pA)) = \text{Hom}(Q/Z, D) \oplus \text{Hom}(Q/Z, \prod_{p \in P}(A/Ap))$ The first summand is 0, since Q/Z is torsion and D, the divisible hull of a torsion free group, is torsion free. Furthermore, $\text{Hom}(Q/Z, \prod_{p \in P}(A/pA)) \cong \prod_{p \in P} \text{Hom}(Q/Z, A/pA) = \prod_{p \in P} \text{Hom}(Q/Z, Z(p)) = 0$ since Q/Z is divisible and $Z(p)$, the cyclic group of order p, is reduced. Also the last group

$$\text{Next}(Q/Z, D \oplus \prod_{p \in P}(A/pA)) = \text{Next}(Q/Z, D) \oplus \text{Next}(Q/Z, \prod_{p \in P}(A/pA)$$

The first summand is 0, since D is a divisible group. Also

$$\text{Next}(Q/Z, \prod_{p \in P}(A/pA)) \cong \prod_{p \in P} \text{Next}(Q/Z, A/pA) = \prod_{p \in P} \text{Next}(Q/Z, Z(p)) = 0$$

since $Z(p)$ is an elementary p-group (see [4]) Thus

$$\text{Next}(Q/Z, A) \cong \text{Hom}(Q/Z, (D \oplus \prod_{p \in P}(A/pA))/gA)$$

Since Q/Z is a torsion group, $\text{Hom}(Q/Z, (D \oplus \prod_{p \in P}(A/pA))/gA)$, and hence $\text{Next}(Q/Z, A)$ is reduced algebraically compact group (see Theorem 46.1 of [1])

Concerning Hext_p we prove that the pure-high extensions of a torsion free group by Q/Z, split, and hence a torsion free group is a $H_p^{\prime}$-group. (see [4]).

THEOREM 2. *For a torsion free group A,* $\text{Hext}_p(Q/Z, A) = 0$.

PROOF. $\text{Hext}_p(Q/Z, A) = \bigcap_{p \in P} p \, \text{Pext}(Q/Z, A)$, (see theorem 7 of [2]) and $\text{Pext}(Q/Z, A)$ is reduced (see lemma 55.3 of [1]). On the other hand $\text{Ext}(Q/Z, A)$ is algebraically compact, and its first Ulm subgroup $\text{Pext}(Q/Z, A)$ must be divisible(see exercise 7 Page 162 of [1]). Hence $\text{Pext}(Q/Z, A)$ and therefore, $\text{Hext}_p(Q/Z, A) = 0$

Now we study the nature of the Frattini subgroups of $\text{Ext}(Q/Z, A)$ and $\text{Pext}(Q/Z, A)$ when A is a torsion group.

THEOREM 3. *Let G_t be the torsion part of G, then*
$\text{Next}(Q/Z, G) \cong \text{Next}(Q/Z, G_t) \oplus \text{Next}(Q/Z, G/G_t)$.
Hence $\text{Next}(Q/Z, G_t)$ *is algebraically compact, whenever* $\text{Next}(Q/Z, G)$ *is.*

PROOF. Since the torsion part G_t of the group G is a neat subgroup of G, the sequence, (with the notation $G/G_t=F$)

$$0\longrightarrow G_t\longrightarrow G\longrightarrow F\longrightarrow 0$$

is neat exact and yields the exact sequence

$$\mathrm{Hom}(Q/Z,\ F)\longrightarrow \mathrm{Next}(Q/Z,\ G_t)\longrightarrow \mathrm{Next}(Q/Z,\ G)\longrightarrow \mathrm{Next}(Q/Z,\ F)\longrightarrow 0$$

The first group is 0, since Q/Z is torsion and $F=G/G_t$ is torsion free. If α is a monomorphism of F into $D\oplus\prod_{p\in P}(F/pF)$, where D is the divisible hull of F, (as defined in theorem 1) then (by theorem 1 and example 2 Page 43 of [1]), we have

$$\begin{aligned}\mathrm{Next}(Q/Z,\ F)&\cong\mathrm{Hom}(Q/Z,\ (D\oplus\prod_{p\in P}(F/pF))/\alpha F)\\ &\cong\mathrm{Hom}(\bigoplus_{p\in P}Z(p^{\infty}),\ (D\oplus\prod_{p\in P}(F/pF))/\alpha F)\\ &\cong\prod_{p\in P}(\mathrm{Hom}(Z(p^{\infty}),\ (D\oplus\prod_{p\in P}(F/pF))/\alpha F)\end{aligned}$$

But because of theorem 44.3 of [1] these products are torsion free, hence $\mathrm{Next}(Q/Z,\ F)$ stays torsion free. Furthermore, (by lemma 1) $\mathrm{Next}(Q/Z,\ G_t)$ is cotorsion, hence the sequence

$$0\longrightarrow\mathrm{Next}(Q/Z,\ G_t)\longrightarrow\mathrm{Next}(Q/Z,\ G)\longrightarrow\mathrm{Next}(Q/Z,\ F)\longrightarrow 0$$

splits and

$$\mathrm{Next}(Q/Z,\ G)\cong\mathrm{Next}(Q/Z,\ G_t)\oplus\mathrm{Next}(Q/Z, G/G_t).$$

Since every direct summand is a pure subgroup it follows that the splitting sequence is pure exact. Also a direct summand of an algebraically compact group is algebraically compact it follows $\mathrm{Next}(Q/Z,\ G_t)$ is algebraically compact whenever $\mathrm{Next}(Q/Z,\ G)$ is.

In case of pure-high extensions this theorem takes the form

THEOREM 4. *If G_t is the torsion part of G, then*

$$\mathrm{Hext}_p(Q/Z,\ G)\cong\mathrm{Hext}_p(Q/Z,\ G_t)$$

PROOF. The proof is much the same as that of theorem 3. To have more insight in the Frattini subgroups of Ext and Pext we discuss a usefull exact sequence, which is contained in

THEOREM 5. *If D is the divisible part of G then the exact sequence $O\longrightarrow D\longrightarrow G\longrightarrow G/D\longrightarrow O$ yields the exact sequence*

$$0 \longrightarrow \mathrm{Next}(G/D, \underset{p\in P}{\oplus} Z(p)) \longrightarrow \mathrm{Next}(G, \underset{p\in P}{\oplus} Z(p))$$
$$\longrightarrow \mathrm{Hom}(D, \prod_{p\in P} Z(p)/ \underset{p\in P}{\oplus} Z(p)) \longrightarrow O$$

where $Z(p)$ stands for cyclic group of order p.

PROOF. Theorem 44.4 of [1] implies that the exact sequence

$0 \longrightarrow D \longrightarrow G \longrightarrow G/D \longrightarrow 0$ induces the exact sequence

$$0 \longrightarrow \mathrm{Hom}(G/D, \prod_{p\in P} Z(p)) \longrightarrow \mathrm{Hom}(G, \prod_{p\in P} Z(p)) \longrightarrow \mathrm{Hom}(D, \prod_{p\in P} Z(p))$$

But, $\mathrm{Hom}(D, \prod_{p\in P} Z(p)) \cong \prod_{p\in P} \mathrm{Hom}(D, Z(p)) = 0$, since D is divisible and $Z(p)$ is reduced. We obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}(G/D, \prod_{p\in P} Z(p)) \longrightarrow \mathrm{Hom}(G, \prod_{p\in P} Z(p)) \longrightarrow 0$$

From theorem 9.2 and exercise 9.14 of [5] we know that $\underset{p\in P}{\oplus} Z(p)$ coincides with the maximal torsion subgroup of $\prod_{p\in P} Z(p)$ and the factor group $\prod_{p\in P} Z(p)/\underset{p\in P}{\oplus} Z(p)$ is divisible. It follows from theorem 44.5. of [1] that the sequence

$$0 \longrightarrow \mathrm{Hom}(G/D, \prod_{p\in P} Z(p)/\underset{p\in P}{\oplus} Z(p)) \longrightarrow \mathrm{Hom}(G, \prod_{p\in P} Z(p)/\underset{p\in P}{\oplus} Z(p))$$
$$\longrightarrow \mathrm{Hom}(D, \prod_{p\in P} Z(p)/\underset{p\in P}{\oplus} Z(p)) \longrightarrow 0$$

is exact. Also the neat exact sequence

$$0 \longrightarrow \underset{p\in P}{\oplus} Z(p) \longrightarrow \prod_{p\in P} Z(p) \longrightarrow \prod_{p\in P} Z(p)/\underset{p\in P}{\oplus} Z(p) \longrightarrow 0$$

yields the exact sequence

$$\mathrm{Hom}(G/D, \prod_{p\in P} Z(p)) \longrightarrow \mathrm{Hom}(G/D, \prod_{p\in P} Z(p)/\underset{p\in P}{\oplus} Z(p)$$
$$\longrightarrow \mathrm{Next}(G/D, \underset{p\in P}{\oplus} Z(p) \longrightarrow \mathrm{Next}(G/D, \prod_{p\in P} Z(p))$$

and

$$\mathrm{Hom}(G, \prod_{p\in P} Z(p)) \longrightarrow \mathrm{Hom}(G, \prod_{p\in P} Z(p)/\underset{p\in P}{\oplus} Z(p))$$
$$\longrightarrow \mathrm{Next}(G, \underset{p\in P}{\oplus} Z(p)) \longrightarrow \mathrm{Next}(G, \prod_{p\in P} Z(p))$$

Since, $\mathrm{Next}(G/D, \prod_{p\in P} Z(p)) \cong \prod_{p\in P} \mathrm{Next}(G/D, Z(p)) = 0$ Also, $\mathrm{Next}(G, \prod_{p\in P} Z(p)) = 0$ because $Z(p)$ is an elementary p-group. The short exact sequences discussed above yields the following commutive diagram.

$$\begin{array}{ccccccc}
0\longrightarrow \operatorname{Hom}(G/D, \prod_{p\in P} Z(p)) & \xrightarrow{f_1} & \operatorname{Hom}(G, \prod_{p\in P} Z(p)) & \longrightarrow 0 & & \\
\Big\downarrow g_1 & & \Big\downarrow g_2 & & & \\
0\longrightarrow \operatorname{Hom}(G/D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)) & \xrightarrow{f_2} & \operatorname{Hom}(G, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)) & \longrightarrow & \operatorname{Hom}(D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)) \longrightarrow 0 \\
\Big\downarrow & & \Big\downarrow & & & \\
\operatorname{Next}(G/D, \bigoplus_{p\in P} Z(p)) & \longrightarrow & \operatorname{Next}(G, \bigoplus_{p\in P} Z(p)) & & & \\
\Big\downarrow & & \Big\downarrow & & & \\
0 & & 0 & & &
\end{array}$$

Since, $\operatorname{Next}(G, \bigoplus_{p\in P} Z(p))$ and $\operatorname{Hom}(D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$ being epimorphic images of $\operatorname{Hom}(G, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$ with kernals $\operatorname{Im} g_2$ and $\operatorname{Im} f_2$. Also we have $\operatorname{Im} g_2 = \operatorname{Im} g_2 f_1 = \operatorname{Im} f_2 g_1 \subseteq \operatorname{Im} f_2$, the third row can be extended to $\longrightarrow \operatorname{Hom}(D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$. Now the three column and first two rows in the commutative diagram are exact, it follows by 3×3 lemma that the third row is exact. We obtain the exact sequence

$$0\longrightarrow \operatorname{Next}(G/D, \bigoplus_{p\in P} Z(p)) \longrightarrow \operatorname{Next}(G, \bigoplus_{p\in P} Z(p)) \longrightarrow \operatorname{Hom}(D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)) \longrightarrow 0$$

as desired.

An analogous theorem for pure-high extensions which can be proved on the same lines is as follows:

THEOREM 6. *If D is the divisible part of G, the exact sequence $0\longrightarrow D\longrightarrow G \longrightarrow G/D\longrightarrow 0$ yields the exact sequence*

$$0\longrightarrow \operatorname{Hext}_p(G/D, \bigoplus_{p\in P} Z(p)) \longrightarrow \operatorname{Hext}_p(G, \bigoplus_{p\in P} Z(p)) \longrightarrow \operatorname{Hom}(D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)) \longrightarrow 0$$

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