Kyungpook Math. J.
Volume 21. Number 1
June. 1981

PRINCIPAL SOLUTIONS OF 2N-ORDER REAL SELF-ADJOINT DIFFERENTIAL SYSTEMS

By Sung J. Lee

Hartman [3], [4] gave explicit formulas for principal solutions for a second order system of differential equations which is equivalent to a case of identically normal Hamiltonian system. In this note we show that his construction can be carried over to an even order real self-adjoint system of differential equations.

1. Let A, B, C be $n\times n$ continuous matrix-valued function of t on an interval $[a, b)$ ($|a|<\infty$, may be ∞) such that $A^*(t)=A(t)$, $C(t)=C^*(t)$ for $t\in[a, b)$. Consider a Hamiltonian system

$$\begin{cases} y'=B(t)y+C(t)z, \\ z'=-A(t)y-B^*(t)z \end{cases} \tag{1.1}$$

and a matrix equation

$$\begin{cases} Y'=B(t)Y+C(t)Z, \\ Z'=-A(t)Y-B^*(t)Z \end{cases} \tag{1.2}$$

where y, z are $n\times 1$ vectors, and Y, Z are $n\times n$ matrices defined on $[a, b)$. An $n\times n$ matrix solution (Y, Z) of (1.2) is called an anti-principal (non-principal in [3]) if it is conjoined (self-conjugate in [4], isotropic in [2]), $\det Y(t)\neq 0$ for all $t\in[c,b)$ for some $a\leq c<b$, and

$$\lim_{t\to b}\int_c^t Y^{-1}(s)C(s)Y^{*-1}(s)\,ds$$

converges entry-wise to a finite limit. A solution (Y, Z) of (1.2) is called a principal solution if it is conjoined, $\det Y(t)\neq 0$ for all $t\in[c,b)$ for some $a\leq c<b$, and

$$\lim_{t\to b}\left[\int_c^t Y^{-1}(s)C(s)Y^{*-1}(s)ds\right]^{-1}=0.$$

We assume here that (H1) $C(t)$ is non-negative definite on $[a, b)$, (H2) Normality condition, that is, if (y, z) is a solution of (1.1) such that $y(t)\equiv 0$ on some subinterval J of $[a, b)$, then $z(t)\equiv 0$ on J, (H3) there exists a conjoined solution (Y,Z) of (1.2) such that $Y(t)$ is invertible for all t in a neighborhood of b, that is, $\det Y(t)\neq 0$ for all $t\in[c, b)$ for some $a\leq c<b$. When (H1)—(H3) are

satisfied, it is known (Theorem 3, Ch. 2, [2]) that principal solutions of (1. 2) exist. But the construction in [2] involves a limiting process. The aim of this note is to give a simple explicit construction of principal solutions of (1.2) without a limiting process, and apply the result to even order self-adjoint differential systems. Such a construction was given in [4] (See the proof of Theorem 10.5 and §11) in the special case when

$$B=-ER-F^*N,\ C=E \text{ and } A=Q+R^*ER+R^*F^*N+N^*FR+N^*GN$$

where E, F, G, N, R, Q are continuous matrix-valued functions such that $E=E^*$, $G=G^*$, $Q=Q^*$, E is non-negative definite and $\begin{pmatrix} E & F^* \\ F & G \end{pmatrix}$ is non-singular. The normality condition for the corresponding equation (1.1) was assumed.

THEOREM 1. *Suppose that (Y, Z) is a conjoined solution of* (1.2) *such that $\det Y(t)\neq 0$ for all $t\in[c, b)$ for some $a\leq c<b$.*

(i) *Define (Y_1, Z_1) on $[c,b)$ by*

$$\begin{cases} Y_1(t)=Y(t)\left[I_n+\int_c^t Y^{-1}(s)C(s)Y^{*-1}(s)\,ds\right], \\ Z_1(t)=Z(t)\left[I_n+\int_c^t Y^{-1}(s)C(s)Y^{*-1}(s)\,ds\right]+Y^{*-1}(t). \end{cases}$$

Then (Y_1, Z_1) is an anti-principal solution of (1.2).

(ii) *Define (Y_2, Z_2) on $[c, b)$ by*

$$\begin{cases} Y_2(t)=Y_1(t)\int_c^t Y_1^{-1}(s)C(s)Y^{*-1}_1(s)\,ds, \\ Z_2(t)=Z_1(t)\int_c^t Y_1^{-1}(s)C(s)Y^{*-1}_1(s)\,ds-Y^{*-1}_1(t). \end{cases}$$

Then (Y_2, Z_2) is a principal solution of (1. 2).

PROOF. (i) It is clear that $\det Y_1(t)\neq 0$, $c\leq t<b$, and (Y_1, Z_1) is a conjoined solution of (1.2) (proposition 1, p.35, [2]). By (Proposition 3 p.39, [2]) that

$$Y(t)=Y_1(I_n-S_1(t)),\quad (c\leq t<b),$$

where

$$S_1(t)=\int_c^t Y_1^{-1}(s)C(s)Y^{*-1}_1(s)\,ds.$$

Let $r(t)=\max\{|\lambda| : \lambda$ is an eigenvalue of $S_1(t)\}$. Then since $S_1(t)$ is self-adjoint, it follows from a well-known theorem on spectral radius that

$$r(t)=\max\ \{\eta^*S_1(t)\eta : \eta^*\eta=1\}=\|S_1(t)\|, \tag{1.3}$$

where $\|S_1(t)\|$ denotes the norm of $S_1(t)$ when it is considered as an operator from

C^n into C^n. Then by (1.3)

$$r(t)\leq r(t_2),\ (c\leq t_1<t_2<b),$$

and so $\lim_{t\to b} r(t)<\infty$ if, and only if the entries of $S_1(t)$ converges as $t\to b$. To complete the proof, it is sufficient to show that $\lim_{t\to b} r(t)<\infty$. Now it follows from (line 11, p.40, [2]) that

$$I_n=(I_n+S_0(t))(I_n-S_1(t)),\ (c\leq t<b), \tag{1.4}$$

where

$$S_0(t)=\int_c^t Y^{-1}(s)C(s)Y^{*-1}(s)\ ds.$$

Since $I_n-S_1(t)$ is positive definite by (1.4), any eigenvalue of $I_n-S_1(t)$ is less than 1. Hence

$$\lim_{t\to b} r(t)<\infty$$

(ii) By (i), $\lim_{t\to b} S_1(t)$ exists. Let us denote this limit by D. Since $\det Y_1(t)\neq 0$ for all $c\leq t<b$ and (H2) holds, it follows from (Proposition 2, p.38, [2]) that $S_1(t)$ is an increasing function of $t\in[c,b)$. In particular, $\det D\neq 0$, $\det Y_2(t)\neq 0$ for $c\leq t<b$. Let us write

$$\begin{cases} Y_2(t)=Y_1(t)\ (D-S_1(t)) \\ Z_2(t)=Z_1(t)(D-S_1(t))-Y^{*}{}_1^{-1}(t) \end{cases} \tag{1.5}$$

for $c\leq t<b$. Clearly $(Y_2,\ Z_2)$ is a conjoined solution of (1.2). By (Proposition 3, p.39, [2]),

$$Y_1(t)=Y_2(t)(D^{-1}+S_2(t)),\ (c\leq t<b) \tag{1.6}$$

where

$$S_2(t)=\int_c^t Y_2^{-1}(s)C(s)Y^{*}{}_2^{-1}(s)\ ds,\ (c\leq t<b).$$

Thus from (1.5), (1.6) together with (line 11, p.40, [2]),

$$I_n=\left(\int_c^t Y_1^{-1}(s)C(s)Y^{*}{}_1^{-1}(s)\ ds\right)(D^{-1}+S_2(t)),\ (c\leq t\leq b). \tag{1.7}$$

This implies that

$$\lim_{t\to b} S_2^{-1}(t)=0.$$

This completes the proof.

We will say that (1.1) is disconjugate near b if there exists $c\in(a,\ b)$ such that (1.1) is disconjugate on $[c,\ b)$. We note that if (H1) and (H2) are satisfied, then (1.1) is disconjugate near b if, and only if (H3) holds. For the "if" part, see (Theorem 2, p. 39, [2]). The "only if" part was proved in [4] in a special case when C is invertible. For completeness, we will prove "only if" part. (See also the proof of Theorem 10.2, [4] where B corresponds to C).

Let
$$\begin{pmatrix} Y(t) & Y_0(t) \\ Z(t) & Z_0(t) \end{pmatrix}$$

be the $2n\times 2n$ fundamental matrix solution of (1.2) such that $Y(c)=I_n=Z_0(t)$, $Z(c)=Y_0(c)=0$ where (1.1) is assumed disconjugate on $[c, b)$. Then clearly (Y_0, Z_0) is conjoined. We claim that $\det Y_0(t)\neq 0$ for all $t\in(c, b)$. If $\det Y_0(t_1)=0$ for some $c<t_1<b$, then $Y_0(t)\eta=0$ for some non-zero constant vector η. Define $x(t)=Y_0(t)\eta$, $Z(t)=Z_0(t)\eta$. Then (x, y) is a solution of (1.1) such that $x(c)=0=x(t_1)$. Thus $x\equiv 0$. By (H_2), $z(t)\equiv 0$. This means that $z(c)=Z_0(c)\eta=\eta=0$. This is a contradiction.

REMARK 1.1. By (H1), (H2) and (H3), Z_1 and Z_2 in Theorem 1 are determined by Y_1 and Y_2 respectively (see p. 386, [4]).

2. Consider an $2n$ order real self-adjoint system

$$\tau y=\sum_{k=0}^{n} (P_k(t)y^{(k)})^{(k)}=0_{m\times 1}, \tag{2.1}$$

and a $m\times mn$ equation

$$\tau y=\sum_{k=0}^{n}(P_k(t)Y^{(k)})^{(k)}=0_{m\times mn} \tag{2.2}$$

Where y and Y are $m\times 1$ and $m\times mn$ matrices. Here $P_k (0\leq k\leq n)$ are $m\times m$ real hermitian k-times continuously differentiable matrix-valued functions on $[a, b)$ ($|a|<\infty$, b may be infinite) such that $(-1)^n P_n(t)$ is positive definite for all $t\in[a, b)$.

If U is a $m\times r$ matrix, then define $mn\times r$ matrices $u(U)$ and $\zeta(U)$ by

$$u(U)=\begin{pmatrix} U \\ U' \\ \cdot \\ \cdot \\ \cdot \\ U^{(n-1)} \end{pmatrix} \qquad \zeta(U)=\begin{pmatrix} \zeta_1(U) \\ \cdot \\ \cdot \\ \cdot \\ \zeta_n(U) \end{pmatrix}$$

where $\zeta_j(U)$ $(1\leq j\leq n)$ is the $m\times r$ matrix defined by

$$\zeta_j(U)=(-1)^j\sum_{k=j}^{n} (P_k(t)U^{(k)})^{(k-j)}.$$

We can check that if Y, Z are sufficiently differentiable $m\times mn$ matrices, then

$\int_{t_1}^{t_2}(Z^*\tau Y-(\tau Z)^*Y)dt=(\zeta^*(Z)u(Y)-u^*(Z)\zeta(Y))(t_1)-(\zeta^*(Z)u(Y)-u^*(Z)\zeta(Y))(t_2)$ for all $a\leq t_1<t_2<b$.

We say that a $m\times mn$ matrix Y of (2.2) is *anti-principal* if (i) it is conjoined, that is, $\zeta^*(Y)u(Y)$ is hermitian, (ii) $\det u(Y(t))\neq 0$ for all $t\in[c,b)$ for some $a\leq c<b$,

(iii) $\lim_{t\to b}\int_c^t(u(Y(s)))^{-1}C(t)(u^*(Y(s)))^{-1}\,ds$ exists entry-wise.

The Y is called *principal* if (i) is conjoined, $\det u(Y(t))\neq 0$ for all $t\in[c,b)$ for some $a\leq c<b$, and

$$\lim_{t\to b}\left[\int_c^t(u(Y(s)))^{-1}C(s)(u(Y(s)))^{*-1}\,ds\right]^{-1}=0_{mn\times mn},$$

where $C(t)$ $(a\leq c<b)$ is the $mn\times mn$ matrix defined by

$$C(t)=\mathrm{diag}(0_{m\times m},\cdots,0_{m\times m},(-1)^nP_n^{-1}(t)).$$

THEOREM 2. *Assume that X is a $m\times mn$ conjoined solution of* (2.2) *such that* $\det u(X(t))\neq 0$ *for all $t\in[c,b)$ for some $a\leq c<b$.*

(i) *Define a $m\times mn$ matrix X_1 on $[c,b)$ by*

$$u(X_1(t))=u(X(t)\left[I_{nm}+\int_c^t(u(X(s)))^{-1}C(s)(u(X(s)))^{*-1}ds\right]).$$

Then X_1 is an anti-principal solution of (2.2).

(ii) *Let X_1 be as the above. Define a $m\times mn$ matrix X_2 on $[c,b)$ by*

$$u(X_2(t))=u(X_1(t))\int_c^t(u(X_1(s)))^{-1}C(s)(u(X_1(s)))^{*-1}\,ds.$$

Then X_2 is a principal solution of (2.2).

PROOF. Define $mn\times mn$ matrices A,B on $[a,b)$ by

$$A=-\mathrm{diag}(P_0,\cdots,(-1)^kP_k,\cdots,(-1)^{n-1}P_{n-1}),$$

$$B=\begin{pmatrix}0_{r\times m} & I_r\\ 0_{m\times m} & 0_{m\times r}\end{pmatrix},$$

where $r=(n-1)m$. Then $A=A^*$.

We can check easily (cf. p.76, [2]) that (2.1) is equivalent to

$$\begin{cases}u'(y)=B(t)\ u(y)+C(t)\ \zeta(y),\\ \zeta(y)=-A(t)\ u(y)-B^*(t)\zeta(y),\end{cases}\tag{2.3}$$

where $C(t)$ is defined earlier, which is non-negative definite. This is identically normal. Thus the result follows from Theorem 1 and Remark 1.1.

COROLLARY 3. *Assume $n=1$. Suppose that X is a $m\times m$ matrix solution of* (2.2) *such that* $\det X(t)\neq 0$ *for all $t\in[c, b)$ for some $a\leq c<b$. Then we have the following*:

(i) *Define a $m\times m$ matrix X_1 on $[c,b)$ by*

$$X_1(t)=X(t)\left[I_m-\int_c^t X^{-1}(s)\, P_1^{-1}(s)X^{*-1}(s)\, ds\right].$$

Then X_1 is an anti-principal solution of (2.2).

(ii) *Let X_1 be as the above. Define a $m\times m$ matrix X_2 on $[c, b)$ by*

$$X_2(t)=-X_1(t)\int_t^b X_1^{-1}(s)\, P_1^{-1}(s)X_1^{*-1}(s)\, ds.$$

Then X_2 is a principal solution of (2.2).

REMARK. The above Corollary was obtained in [3], and later in [4] in a more general second order system.

Department of Mathematics
Pan American University
Edinburg, TX 78539

REFERENCES

[1] Ahlbrandt, C.D., *Principal solutions of self-adjoint differential systems and their reciprocals*, Rocky Mountain J. Math., Vol. 2, No. 2(1972), 169—182.

[2] Coppel, W.A., *Disconjugacy*, Lecture notes in Mathematics, Vol. 220, Springer-Verlag (1971), Berlin, Heidelberg, New York.

[3] Hartman, P., *Self-adjoint, non-oscillatory systems of ordinary second order, linear differential equations*, Duke Math. J. (24) (1957), 25—35.

[4] Hartman, P., *Ordinary differntial equations*, John Wiley (1964), New York.

[5] Naimark, M.A., *Linear differential operators*, Part I, Frederick Ungar (1967), New York.

[6] Reid, W.T., *Principal solution of non-oscillatory self-adjoint linear differential systems*, Pacific J. Math., (80) (1958), 147—169.

[7] Reid, W.T., *Riccati matrix differential equations and non-oscillatory criteria for associated linear differential systems*, Pacific J. Math., (13) (1963), 665—685.

[8] Reid, W.T., *Ordinary differential equations*, Wiley-Interscience (1971) New York.