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# AN ANALYTIC SUFFICIENCY CONDITION FOR GOLDBACH'S CONJECTURE WITH MINIMAL REDUNDANCY II

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This paper is a continuation of [1], and the notation here is the same as in that paper.

By a straightforward computation it can be shown that

$$r(n)=4\int_{x_0}^{x_0+1} f_c^2(x,\ n)\cos(2\pi nx)dx=4\int_{x_0}^{x_0+1} -f_s^2(x,\ n)\cos(2\pi nx)\,dx$$

for any $x_0$; so that

$$r(n)=4\int_{m(n)} f_c^2(x,\ n)\cos(2\pi nx)dx+4\int_{M(n)} f_c^2(x,\ n)\ \cos(2\pi nx)dx\,;$$

$$r(n)=4\int_{m(n)} -f_s^2(x,\ n)\cos(2\pi nx)dx+4\int_{M(n)} -f_s^2(x,\ n)\cos(2\pi nx)\,dx.$$

We establish the following

THEOREM 1. *If either*

$$\int_{m(n)} f_c^2(x,\ n)\cos(2\pi nx)\ dx=0(n\ \log^{-2} n)$$

*or*

$$\int_{m(n)} f_s^2(x,\ n)\ \cos(2\pi nx)dx=0(n\ \log^{-2} n),$$

*then* $r(n)>0$ *for every even* $n\geq N_0$.

PROOF. Fix $n\geq N_0$. By definition

$$4\int_{M(n)} f_c^2(x,\ n)\cos(2\pi nx)dx=4\sum_{\substack{q\leq \log^{15}n\\ (q,\ n)=1}}\ \sum_{\substack{0<h\leq q\\ (h,q)=1}} T_c(h,q)$$

$$4\int_{M(n)} -f_s^2(x,\ n)\cos(2\pi nx)\,dx=4\sum_{\substack{q\leq \log^{15}n\\ (q,\ n)=1}}\ \sum_{\substack{0<h\leq q\\ (h,q)=1}} T_s(h,\ q)$$

where

$$T_c(h,\ q)=\int_{\frac{h}{q}-x_0}^{\frac{h}{q}+x_0} f_c^2(x,\ n)\cos(2\pi nx)\,dx$$

and

$$T_s(h,\ q)=\int_{\frac{h}{q}-x_0}^{\frac{h}{q}+x_0} -f_s^2(x,\ n)\cos\ (2\pi nx)\ dx$$

By the same argument as presented in [1] we have if $(h,\ q)=1$ and $q\leq\log^{15}n$, then

$$\Big|T_c(h,\ q)-\frac{\mu^2(q)}{\phi^2(q)}\Big[\cos\Big(2\pi n\frac{h}{q}\Big)\int_{-x_0}^{x_0} g_c^2(y,\ n)\cos(2\pi ny)\,dy\Big]$$

$$+\frac{\mu^2(q)}{\phi^2(q)}\Big[\sin\Big(2\pi n\frac{h}{q}\Big)\int_{-x_0}^{x_0} g_c^2\ (y,\ n)\ \sin\ (2\pi ny)\ dy\Big]\Big|$$

$$\leq 8n\log^{-54}n\,; \tag{A}$$

and

$$\Big|T_s(h,\ q)-\frac{\mu^2(q)}{\phi^2(q)}\Big[\cos\Big(2\pi n\frac{h}{q}\Big)\int_{-x_0}^{x_0} -g_s^2\ (y,\ n)\cos(2\pi ny)\ dy\Big]$$

$$+\frac{\mu^2(q)}{\phi^2(q)}\Big[\sin\Big(2\pi n\frac{h}{q}\Big)\int_{-x_0}^{x_0} -g_s^2\ (y,\ n)\sin(2\pi ny)\ dy\Big]\Big|$$

$$\leq 8n\log^{-54}n. \tag{B}$$

Let

$$T_c(n)=\int_{-x_0}^{x_0} g_c^2\ (y,\ n)\cos(2\pi ny)\,dy,$$

$$T_c^*(n)=\int_{-x_0}^{x_0} g_c^2\ (y,\ n)\sin(2\pi ny)\ dy,$$

$$T_s(n)=\int_{-x_0}^{x_0} -g_s^2\ (y,\ n)\cos(2\pi ny)\ dy,$$

$$T_s^*(n)=\int_{-x_0}^{x_0} -g_s^2(y,\ n)\ \sin(2\pi ny)\ dy.$$

Let

$$T(n)=\frac{1}{4}\sum_{m_1,\, m_2}(\log^{-1} m_1)(\log^{-1} m_2)\;;$$

with the conditions of summation $m_1 \geq 2$, $m_2 \geq 2$, and $(m_1+m_2)=n$.

By a straightforward computation it can be shown that

$$T(n)=\int_{-1/2}^{1/2} g_c^2(y,\, n)\cos(2\pi ny)\; dy \tag{C}$$

and

$$T(n)=\int_{-1/2}^{1/2} -g_s^2(y,\, n)\cos(2\pi ny)\, dy \tag{D}$$

By the same argument as presented in [1] it can be shown that if $(h,\, q)=1$ and $q \leq \log^{15} n$, then

$$\left|\cos\left(2\pi n\,\frac{h}{q}\right)\right|\;\left|\frac{\mu^2(q)}{\phi^2(q)}\right|\;\left|T(n)-T_c(n)\right| \leq \frac{1}{\phi^2(q)}\,(2n\log^{-15} n) \tag{E}$$

and

$$\left|\cos\left(2\pi n\,\frac{h}{q}\right)\right|\;\left|\frac{\mu^2(q)}{\phi^2(q)}\right|\;\left|T(n)-T_s(n)\right| \leq \frac{1}{\phi^2(q)}\,(2n\log^{-15} n). \tag{F}$$

By (A), (B), (E) and (F) we have that if $(h, q)=1$ and $q \leq \log^{15} n$, then

$$\left|T_c(h,\, q)-\frac{\mu^2(q)}{\phi^2(q)}\cos\left(2\pi n\,\frac{h}{q}\right) T(n)+\frac{\mu^2(q)}{\phi^2(q)}\sin\left(2\pi n\,\frac{h}{q}\right) T_c^*(n)\right|$$
$$\leq 8n\log^{-54} n+\frac{1}{\phi^2(q)}(2n\log^{-15} n)\;;$$

and

$$\left|T_s(h,\, q)-\frac{\mu^2(q)}{\phi^2(q)}\cos\left(2\pi n\,\frac{h}{q}\right) T(n)+\frac{\mu^2(q)}{\phi^2(q)}\sin\left(2\pi n\,\frac{h}{q}\right) T_s^*(n)\right|$$
$$\leq 8n\log^{-54} n+\frac{1}{\phi^2(q)}\,(2n\log^{-15} n).$$

The rest of the proof now follows as in [1].

REMARK. It is easy to see by the prime number theorem that we have at least $0(n\log^{-1} n)$ for both of the integrals in the hypothesis of Theorem 1.

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## BIBLIOGRAPHY

[1] *An analytic sufficiency condition for Goldbach's conjecture with minimal redundancy,* Kyungpook Mathematical Journal Vol. 20, No. 1, 1980, 1—8.