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COMPLETELY COMPACT SPACES

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By a *completely compact space,* we shall mean a space in which every subset is compact.

In this note, we will give a variety of characterizations of completely compact spaces (when T_1 is assumed) and give some properties of completely compact topologies.

We begin with

LEMMA 1. *If $(X, \mathscr{T})$ is an infinite Hausdorff space, there exists an infinite sequence of pairwise disjoint non-empty open subsets of X.*

This is Theorem 5.2.3 in [2].

LEMMA 2. *If $(X, \mathscr{T})$ has an infinite number of components, then there exists an infinite sequence of pairwise disjoint non-empty open subsets of X.*

See Theorem 1 in [1].

THEOREM 3. *Let $(X, \mathscr{T})$ be a T_1-space. Then the following are equivalent:*

(1) *X is completely compact*

(2) *every open subset of X is compact*

(3) *every subset of X is sequentially compact*

(4) *every subset of X is countably compact*

(5) *every countable subset of X is compact*

(6) *every subset of X has a finite number of components*

(7) *$A \cap A'$ is infinite for every infinite subset A of X, A' denoting the derived set of A*

(8) *X contains no infinite discrete subset*

(9) *X contains no infinite Hausdorff subspace.*

PROOF. (1) implies (2). This is clear.

(2) implies (3). Let A be a subset of X and suppose $\{a_n : n \geq 1\}$ is a sequence S_0 in A. Assume no subsequence of S_0 converges to a point of A. Then S_0 does not converge to a_1 and hence there exists an open set O_1 and a subsequence S_1 of

S_0 such that $a_1 \in O_1$ and S_1 lies in $\mathscr{C}O_1$, $\mathscr{C}$ denoting the complement operator. Now S_1 does not converge to a_2 and hence there exists an open set O_2 and a subsequence S_2 of S_1 such that $a_2 \in O_2$ and S_2 lies in $\mathscr{C}O_2 \cap \mathscr{C}O_1$. By induction we have a sequence of open sets O_i and a sequence of sequences S_i such that $a_i \in O_i$, S_{i+1} is a subsequence of S_i and S_i lies in $\mathscr{C}O_i \cap \mathscr{C}O_{i-1} \cap \cdots \cap \mathscr{C}O_1$. By (2), $\bigcup \{O_i : i \geq 1\}$ is compact and hence $\bigcup \{O_i : i \geq 1\} = O_1 \cup \cdots \cup O_N$ for some N. But S_N is in $\bigcup \{O_i : i \geq 1\}$ and hence is in $O_1 \cup \cdots \cup O_N$. However S_N lies in $\mathscr{C}O_N \cap \cdots \cap \mathscr{C}O_1 = \mathscr{C}(O_1 \cup \cdots \cup O_N)$, a contradiction.

(3) implies (4). Let A be a subset of X and suppose $A \subseteq O_1 \cup O_2 \cup \cdots$. Suppose that A is contained in $O_1 \cup \cdots \cup O_n$ for no n. Take $a_1 \notin O_1$, $a_2 \notin O_1 \cup O_2, \cdots, a_n \notin O_1 \cup \cdots \cup O_n, \cdots$. Let $B = \{a_n : n \geq 1\}$. It is clear that no subsequence of $\{a_n : n \geq 1\}$ converges to a point of B. Thus B is not sequentially comapct.

(4) implies (5). Let A be a countable subset of X. Then A is a Lindelof space and by (4) countably compact. Thus A is compact.

(5) implies (6). Let A be a subset of X with an infinite number of components. By Lemma 2, there exist an infinite sequence of non-empty pairwise disjoint sets B_i which are open in A. Let $b_i \in B_i$ for each i. Then $\{b_i : i \geq 1\}$ is a countable subset of X which is not compact.

(6) implies (7). Let A be an infinite subset of X and suppose that $A \cap A'$ is finite. Then $A - A'$ is infinite; take $a_1, a_2, \cdots$ an infinite sequence of distinct points in $A - A'$. For each i, there exists an open set O_i such that $a_i \in O_i$ and $A \cap O_i - a_i = \phi$ or $A \cap O_i = \{a_i\}$. Let $B = \{a_i : i \geq 1\}$. Then B is infinite discrete and hence has an infinite number of components.

(7) implies (8). Suppose $A \subseteq X$ and A is infinite and discrete. Take $a \in A$; there exists an open set O such that $\{a\} = A \cap O$. Then $A \cap O - a = \phi$ and $a \notin A'$. Thus $A \cap A' = \phi$.

(8) implies (9). Suppose $A \subseteq X$, A is infinite and A is a Hausdorff subspace of X. By Lemma 1, there exists a sequence of non-empty disjoint sets A_i which are open in A. Let $a_i \in A_i$ for each i and let $B = \{a_i : i \geq 1\}$. Then B is infinite and discrete.

(9) implies (1). (Here is where T_1 is used.) Suppose $A \subseteq X$ and A is not compact. Then there exists $\{O_\alpha : \alpha \in \Delta\}$, an open cover of A with no finite subcover. Take $a_1 \in A$; then $a_1 \in O_{\alpha_1}$ for some α_1. Take a_2 in A such that $a_2 \notin O_{\alpha_1}$. $a_2 \in O_{\alpha_2}$ for some α_2. By induction there exists sequences $\{a_i : i \geq 1\}$ and $\{\alpha_i : i \geq 1\}$

such that $a_i \in O_{\alpha_i}$ and $a_i \notin O_{\alpha_1} \cup \cdots \cup O_{\alpha_{i-1}}$ for $i \geq 2$. Let $B=\{a_i : i \geq 1\}$. Then B is infinite and Hausdorff. Let $a_n \neq a_m$ and assume that $n<m$. Then $a_n \in B \cap O_{\alpha_n}$ and $a_m \in B \cap (O_{\alpha_m} - \{a_1, \cdots, a_m\})$ and $B \cap O\alpha_n$ and $B \cap (O_{\alpha_m} - \{a_1, \cdots, a_n\})$ are disjoint and open in B.

LEMMA 4. *Let $\mathscr{T}$ and $\mathscr{U}$ be topologies on X for which $(X, \mathscr{T})$ and $(X, \mathscr{U})$ are completely compact. Let $\mathscr{V} = \sup\{\mathscr{T}, \mathscr{U}\}$. Then $(X, \mathscr{V})$ is completely compact.*

PROOF. Let $\mathscr{S} = \mathscr{T} \cup \mathscr{U}$; then $\mathscr{S}$ is a subbase for $\mathscr{V}$. It suffices to show that every subset A of X is $\mathscr{S}$-compact. Let $A \subseteq X$ and $A \subseteq \bigcup\{O_\alpha : \alpha \in \Delta\} \cup \bigcup\{U_\gamma : \gamma \in \Gamma\}$ where $O_\alpha \in \mathscr{T}$ for each $\alpha \in \Delta$ and $U_\gamma \in \mathscr{U}$ for each $\gamma \in \Gamma$. Now $\{O_\alpha : \alpha \in \Delta\}$ is a $\mathscr{T}$-open cover of $\bigcup\{O_\alpha : \alpha \in \Delta\}$ and hence $\bigcup\{O_\alpha : \alpha \in \Delta\} = O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$ for some $\alpha_1, \cdots, \alpha_n$ in Δ. Likewise $\bigcup\{U_\gamma : \gamma \in \Gamma\} = U_{\gamma_1} \cup \cdots \cup U_{\gamma_m}$ for some $\gamma_1, \cdots, \gamma_m$ in Γ. Thus $A \subseteq O_{\alpha_1} \cup \cdots \cup O_{\alpha_n} \cup U_{\gamma_1} \cup \cdots \cup U_{\gamma_m}$.

LEMMA 5. *Let $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ be a surjection and let $\mathscr{T}$ be the weak topology, that is, $\mathscr{T} = \{f^{-1}[U] : U \in \mathscr{U}\}$. If $(Y, \mathscr{U})$ is completely compact, then so is $(X, \mathscr{T})$.*

PROOF. Let $A \subseteq X$ and suppose $A \subseteq \bigcup\{f^{-1}[U_\alpha] : \alpha \in \Delta\}$. Then $f[A] \subseteq \bigcup\{U_\alpha : \alpha \in \Delta\}$ and $f[A]$ is compact. Thus $f[A] \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ for some $\alpha_1, \cdots, \alpha_n$ in Δ. Then $A \subseteq f^{-1}[U_{\alpha_1}] \cup \cdots \cup f^{-1}[U_{\alpha_n}]$.

LEMMA 6. *Let $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ be a continuous surjection and suppose that $(X, \mathscr{T})$ is completely compact. Then $(Y, \mathscr{U})$ is completely compact.*

We omit the easy proof.

THEOREM 7. *Let $(Z, \mathscr{W})$* be the product space of $(X, \mathscr{T})$ and $(Y, \mathscr{U})$. Then $(Z, \mathscr{W})$ is completely compact iff $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ *are completely compact.*

PROOF. If $(Z, \mathscr{W})$ is completely compact, then $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ are completely compact. This follows from Lemma 6.

If $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ are completely compact, then so is $(Z, \mathscr{W})$. This follows from Lemmas 4 and 5.

Theorem 7 cannot be extended to infinite product as is shown by

EXAMPLE 8. Let $Y = \{a, b\}$ and $\mathscr{U} = \{\phi, \{a\}, \{b\}, Y\}$. Let $(X_i, \mathscr{T}_i) = (Y, \mathscr{U})$ for $i \geq 1$ and let $(X, \mathscr{T}) = \prod\{(X_i, \mathscr{T}_i) : i \geq 1\}$. Then $(X_i, \mathscr{T}_i)$ is completely compact for all i, but $(X, \mathscr{T})$ is not. For if every subset of X were compact, then every set would be closed (X is Hausdorff) and $(X, \mathscr{T})$ would be discrete.

THEOREM 9. *Let $(X, \mathscr{T})$ be infinite and completely compact. Then there exists a topology $\mathscr{U}$ on X for which $\mathscr{T} \subseteq \mathscr{U}$, $\mathscr{T} \neq \mathscr{U}$ and $(X, \mathscr{U})$ is completely compact.*

PROOF. $\mathscr{T} \neq \mathscr{P}(X)$ lest $(X, \mathscr{T})$ be discrete and not compact. Let $A \in \mathscr{P}(X) - \mathscr{T}$. Let $\mathscr{V} = \{\phi, A, X\}$. Then $\mathscr{V}$ is a completely compact topology for X; let $\mathscr{U} = \sup\{\mathscr{T}, \mathscr{V}\}$. Then $(X, \mathscr{U})$ is completely compact by Lemma 4.

THEOREM 10. *Let $(X, \mathscr{T})$ be a space which is not completely compact. There exists then a topology $\mathscr{U}$ for X such that $\mathscr{U} \subseteq \mathscr{T}$, $\mathscr{U} \neq \mathscr{T}$ and $(X, \mathscr{U})$ is not completely compact.*

PROOF. Let $A \subseteq X$, A not compact. Let $\{O_\alpha : \alpha \in \Delta\}$ be an open cover of A with no finite subcover. There exists a sequence a_i in A and a sequence α_i in Δ such that $a_i \in O_{\alpha_i}$ for all i and $a_i \notin \alpha_1 \cup \cdots \cup O_{\alpha_{i-1}}$ for $i \geq 2$. Let $\mathscr{U} = \{U \mid U = \phi$ or $U \in \mathscr{T}$ and $U \supseteq O_{\alpha_1} \cup O_{\alpha_2}\}$. Clearly $\mathscr{U}$ is a topology for X, $O_{\alpha_1} \notin \mathscr{U}$ and A is not $\mathscr{U}$-compact.

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REFERENCES

[1] Norman Levine, *American Mathematical Monthly*, Vol. 69, No. 10, December, 1962.

[2] William J. Pervin, *Foundations of General Topology*, Academic Press, New York-London, 1964.