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Comparative Studies of Methods for Continuation and Derivatives of Potential Fields

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Abstract: Studies of model potential fields continued upward and downward show differences depending on the method of continuation. Beginning with a magnetic field computed over a buried vertical cylinder, the field was continued to various levels by a method introduced by Henderson (Lagrangian interpolation) and by a spectral method (frequency domain analysis). Resultant fields show (1) no significant differences in upward continued values, (2) in downward continuation, accurate values are obtained with the spectral method over the central part of the anomaly, and (3) accurate values are obtained with Henderson's method on the flanks of the anomaly, while oscillations usually characterize the spectral method in this region. Essentially the same observations are made for derivative calculations.

Field oscillations are empirically predicted at levels continued to approximately two-thirds of the depth of the source. Our spectral computer program output yields marked oscillations at one-half of the depth of the source. Henderson's method shows no oscillations at this depth and only minor oscillations at the top of the body (some negative values appear on the flanks of the anomaly). The Henderson output is a smooth field even if continued below the top of the body. These results suggest that the presence of oscillations cannot be used to identify the top of a buried source without careful consideration of the method used to continue the field. Use of the derivative to outline and isolate anomalies must similarly include consideration of the method of calculation.

1. Introduction

In principle, a potential field measured at the earth's surface can be used to compute theoretical fields at different levels, above or below the observed field. Upward continuation is sometimes used to suppress local anomalies originating in the near surface. Conversely, downward continuation increases the resolution of weak anomalies. In a similar way, the first and second vertical derivatives of a potential field also can accentuate and isolate weak anomalies originating from shallow sources. Various methods of computing continued fields and their dervatives are summarized in a standard textbook by Grant and West (1965).

Successful continuation of gravity and magnetic fields is based on the assumption that Laplace's equation is not violated. That is, the continued field must be above the disturbing body (mass or magnetized material). If continuation is carried to depths below or near the top of the source, the field will begin to oscillate. Roy(1967) has suggested the oscillation could be a criterion of depth. An important aspect of this study is and evaluation of the behavior of oscillations as the field is continued downward by two different methods: (a) by Lagrangian extrapolation in the space domain as suggested by Henderson(1960) and, (b) by multiplication in the spectral domain following an algorithm by Argawal(1968).

The Henderson and spectral methods are the most popular approaches in computing the derivatives and continued fields of gravity and magnetic data. To date no study has been done empirically comparing the two methods, although they differ significantly. A comparison of these two numerical methods and the reliability of their calculations is the prime purpose of this study.

2. Theory

Henderson's Method: Upward continuation can be calculated from the integral formula of the classic solution of the Dirichlet problem for a half space

$$\Delta\phi(-ma)=\int_0^\infty \frac{ma\overline{\Delta\phi}(r)rdr}{(r^2+m^2a^2)^{3/2}} \qquad (1)$$

where m is an integer, a is the grid interval used to digitize the surface data (Fig. 1),

$$\text{and} \quad \overline{\Delta\phi(r)}=\frac{1}{2\pi}\int_0^{2\pi}\Delta\phi(r,\theta)d\theta \qquad (2)$$

is the average value of $\Delta\phi$ on circles of radius r about the point of calculation. Henderson(1960) calculated upward continued field values, $\Delta\phi$(-ma), and then used these values to extrapolate $\Delta\phi$ downward by the use of the Lagrange interpolation formula

$$\Delta\phi(Z)\simeq\sum_{m=0}^{n}\frac{(-1)^m Z(Z+a)(Z+2a)\cdots(Z+ma)}{a^n(Z+ma)(n-m)!m!}\cdot\Delta\phi(-ma) \qquad (3)$$

Based on empirical studies, he used n=5 as an optimum choice for computation of $\Delta\phi$ at a depth of k grid units below the surface. The general formula is

$$\Delta\phi\simeq\sum_{i=0}^{10}\overline{\Delta\phi}(r_i)D(r_i,\mathrm{k}) \qquad (4)$$

where D($_i$, k) is a fixed set of coefficients. Formula (3) can be rewritten in the following

$$\Delta\phi(Z)\simeq-|\mathrm{V}|^{-1}\begin{vmatrix} 0 & 1 & -Z/a & (Z/a)^2 & \cdot & \cdot & (-Z/a)^5 \\ \Delta\phi(0) & 1 & 0 & 0 & \cdot & \cdot & 0 \\ \Delta\phi(-a) & 1 & 1 & 1 & \cdot & \cdot & 1 \\ \Delta\phi(-2a) & 1 & 2 & 2^2 & \cdot & \cdot & 2^5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta\phi(-5a) & 1 & 5 & 5^2 & \cdot & \cdot & 5^5 \end{vmatrix} \qquad (5)$$

equivalent determinantal form (n=5), where |V| is the Vandermonde determinant obtained by deleting the first two rows and columns of the above determinant.

Since only the first row of the determinant involves the variable Z, the first and second vertical derivatives are obtained by successively differentiating the first row. Both first and second derivatives have the general form of equation (4) with appropriate coefficients $D(r_i,$ k).

Spectral Method: Discretly sampled potential field data at any elevation Z, canbe represented by a double Fourier series(2)

$$\Delta\phi(x,y,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\exp\left\{2\pi Z\sqrt{\frac{m^2}{L_x}+\frac{n^2}{L_y}}\right\}\cdot\left(A_m\cos 2\pi m\frac{x}{L_x}+B_m\sin 2\pi m\frac{x}{L_x}\right)\cdot\left(C_n\cos 2\pi n\frac{y}{L_y}+D_n\sin 2\pi n\frac{y}{L_y}\right), \qquad (6)$$

where L_x, L_y are the fundamental wavelengths in the x and y directions and A_m, B_m, C_n, and D_n are the coefficients in the Fourier series expansion. For an equally spaced square map which contains M grid values along x and y axis, equation 6 can be expressed in a more compact form,

$$\Delta\phi(x,y,z)=\sum_{m=0}^{M-1}\sum_{n=0}^{M-1}F_{mn}e^{(2\pi Z/L)\sqrt{m^2+n^2}}\cdot\frac{\cos}{\sin}m\frac{2\pi x}{L}\frac{\cos}{\sin}n\frac{2\pi y}{L}, \qquad (7)$$

The spectral analysis method involves using the Fast Fourier Transform (FFT)(3) to obtain the coefficients of the map, F_{mn}, and them multiplying these complex numbers by the factor.

$$e^{\frac{2\pi(\pm Z)}{L}\sqrt{m^2+n^2}} \qquad (8)$$

where $+Z$ yields downward continuation and $-Z$ is upward continuation. Computation of the vertical derivatives of k_{th} order is obtained by

Table 1. True and computed values of the magnetic field on the axis of a vertical cylinder(see Fig. 1). Computed values from methods of Henderson (1960) and Argawal (1968): true values from Equation (9).

Depth Z	True Values	Computed values and % Error			
		Henderson		Spectral	
-5	16	17	(6%)	19	(18%)
-4	20	21	(5%)	22	(10%)
-3	26	26	(0%)	28	(7%)
-2	34	35	(3%)	36	(6%)
-1	49	49	(0%)	49	(0%)
0	71	71		71	
1	115	109	(-5%)	112	(-2%)
2	198	175	(-12%)	192	(-3%)
3	378	285	(-25%)	*	

* Oscillatory values—not considered meaningful.

multiplying the coefficients at each level by $\left[\frac{2\pi}{L}\sqrt{m^2+n^2}\right]^k$. The final computation involves transforming the frequency data back to the space domain by inverse Fourier Transform Equation (7).

Convergence of equation 7 depends on the multiplying factor of equation (8). Unless the coefficients F_{mn}[obtained from $\Delta\phi_0(x,y)$] attenuates more rapidly than the exponential term of equation (8), spectral downward continuation is not valied. Therefore, $\Delta\phi_0(x,y)$ should be a smooth function whose Fourier spectrum attenuates with the shorter "wavelength" more rapidly than the exponential term rises. In ordinary applications a high-cut filter can suppress the near-surface (short wavelengths) contribution of the residual potential field. In this study, the fields were generated from a model and no problems of noise are anticipated.

3. Program Development

Henderson's Method: The continuation and derivative program, as developed by Rudman and Blakely (1976), closely follows the algorithm of Henderson (1960) and is entirely in the space domain. The technique involves calculation of an average $\overline{\Delta\phi(r_i)}$ around ten rings of radii r_i centered on each point and then multiplying these values and the surface value by the appropriate coefficients $D(r_i,$ k), where k specifies the continuation level. The coefficients are stored in the computer in a 2 x 2 matrix. Ring values $\Delta\phi(x,y,i)$ are stored in a 3 x 3 matrix; 11 values for each point (x, y). Output is in the form of printed map data.

Spectral Method: The spectral continuation and derivative program, written by the author, follows the algorithm of Argawal (1968) and directly incorporates most of his programming procedures. Before transforming the original map data from the space to frequency domain, the data needs to be arranged in a symmetrical matrix. The need for this arrangement is better understood if one first considers the FFT of 1-dimensional data. In the frequency domain (Fig. 2), the real (R) and imaginary(I) parts of the Fourier coefficients are arranged symmetrically about a central maximum (Nyquist) frequency by th standard FFT program. Because a similar pattern is obtained in a 2-dimensional transform, the original map data f_{ij} must first be reorganized

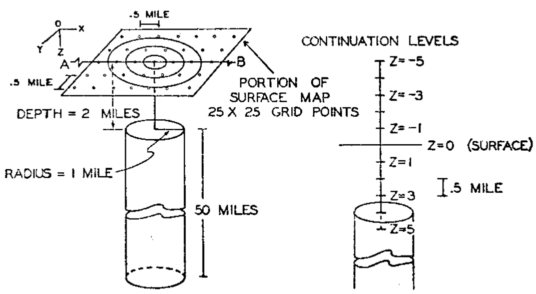

Fig. 1 Model of a long vertical cylinder used to compute a magnetic field at the surface (magnetic susceptibility=.002c.g.s., vertical earth magnetic field=55150 gammas). AB identifies line of cross sections for Figures 4 through 8. Possible levels of continuation sketched relative to position of the cylinder.

into a symmetric matrix (Fig. 3). This arrangement anticipates the standard symmetric FFT separation of F_{mn} coefficients. These F_{nm} values must be multiplied by equation (8), and the m and n values must also be symmetrical (see top and side of Fig. 3). After multiplication to obtain the continued field, the inverse FFT returns the expanded map to the space domain. The program then extracts the original map size for output.

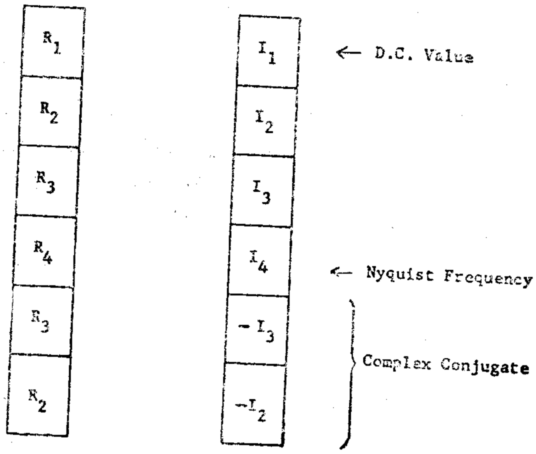

Fig. 2 Sketch illustrating symmetry of 1-dimensional data after Fast Fourier Transform into the frequency domain.

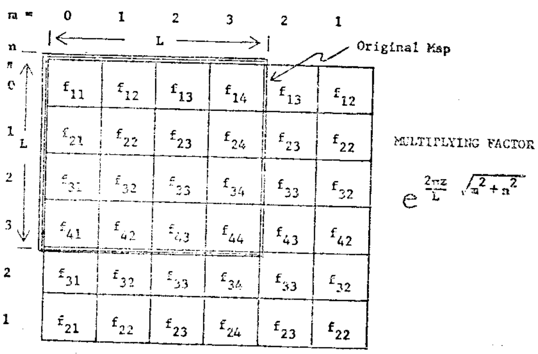

Fig. 3 Sketch illustrating original map data **Fig.** and the expanded symmetrical matrix created before Fast Fourier Transform (FFT). The harmonic coefficients m and n are displayed in the symmetric positions for multiplying the Fourier coefficients after FFT.

4. Empirical Tests and Results

A model magnetic field was computed over a long vertical cylinder at a depth of two miles. The cylinder was one mile in radius and 50 miles in length. The field in gammas was computed at a grid interval of 0.5 mile to yield a 25×25 map output (Fig. 1). Values of this model field were computed from a standard solid angle program (Talwani,et. al., 1960). The 625 values were used as data input for the Henderson and spectral programs. Continued and derivative fields were

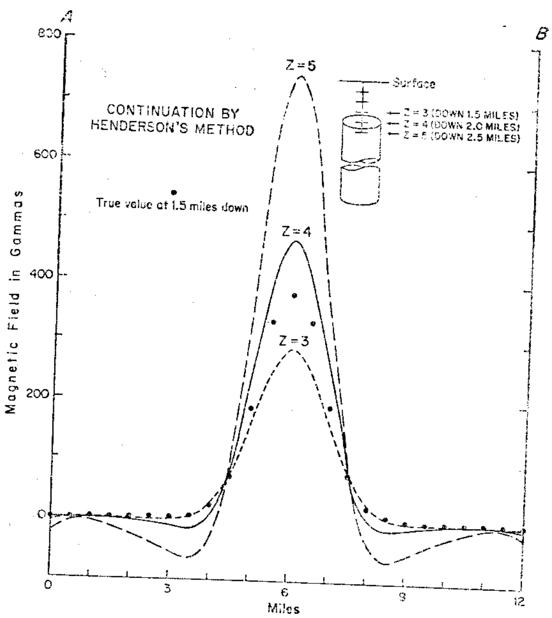

Fig. 6 Cross section of magnetic fields over a cylinder (see Fig. 1) for three levels of continuation using Henderson's method. True values obtained from solid angle calculation.

computed for all levels and the results presented in figures 4 through 8.

Figures 4 and 5 compare the true map values over the cylinder(accepting the solid angle program output as "true values") with continued fields using Henderson(H) and spectral(S)

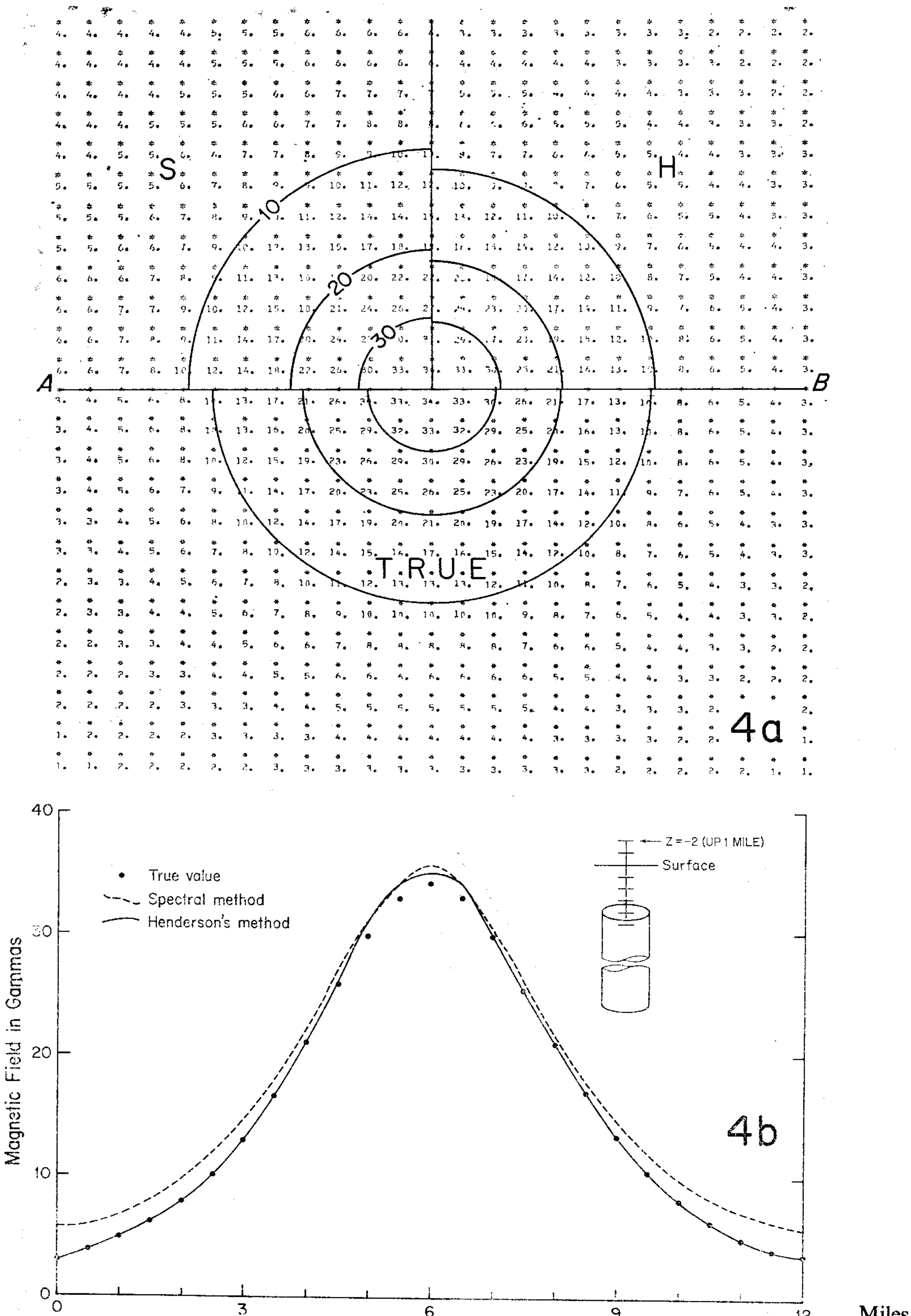

Fig. 4a Map showing the True magnetic field of a vertical cylinder computed from solid angle analysis at a level one mile above the original surface (Fig. 1). The northeast quarter of the map shows the field continued upward one mile (2 grid units) by Henderson's method (H). The northwest quarter is continued upward by the spectral methop (S). Contours in gammas.

4b Cross section along line AB of map showing all three fields.

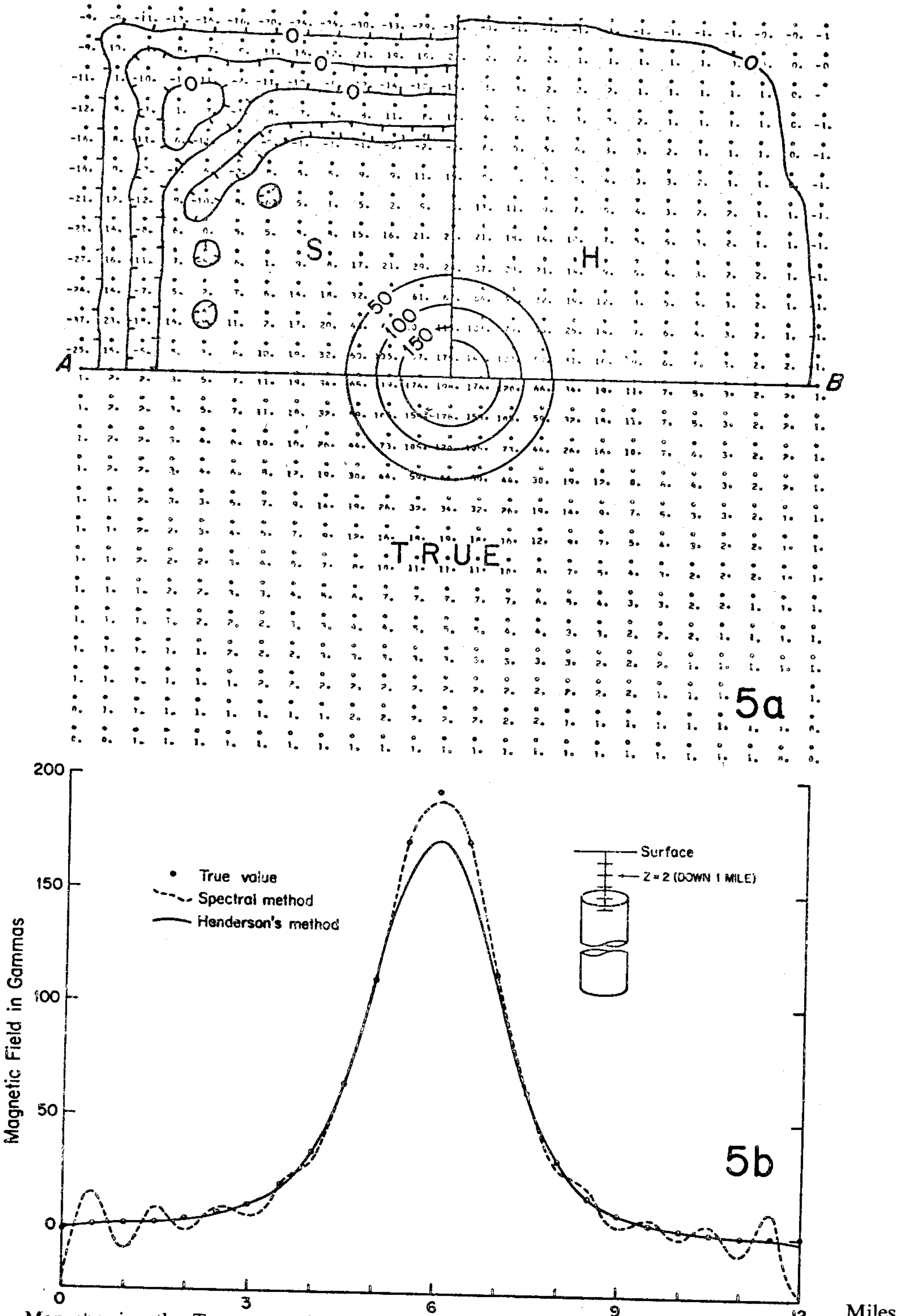

Fig. 5a Map showing the True magnetic field of a vertical cylinder computed from solid angle analysis at a level one mile below the original surface (Fig. 1). The northeast quarter of the map shows the field continued downward one mile by Henderson's method (H). The northwest quarter is continued downward by the spectral method (S). Contours in gammas.

5b Cross section along line AB of map showing all three fields.

methods. For clarity, cross sections are presented along an east-west line AB. For upward continuation of 1.0 mile (2 grid units), the contours are almost equivalent on all three maps (Fig. 4a). In the cross section (Fig. 4b), the true values are very close to both the spectral and Henderson output. Although not shown in this report, upward continuation at all levels is quite successful; errors are minimal, with only a few gammas difference in absolute value.

Figure 5 shows the map and cross section for the field continued downward one mile (2 grid units). The "true" contours over the central part of the anomaly (Fig. 5a) are similar to those computed from the Henderson and spectral approach, with the spectral method yielding slightly better results. However, the spectral map displays strong oscillations along the edges of the map. This is well illustrated in fiigure 5b. Although not shown here, continuation below this one mile level shows the spectral map to be totally unusable. According to Bhattacharya(1965), we expect oscillations at 2/3 of the depth to the body (our results yield oscillations at a somewhat shallower depth).

An unexpected result was observed when the map was continued downward below the top of the body using Henderson's method (Fig.6). According to Roy (1967) we expect oscillatory values to mark the top of the body. Even below the body, the anomaly values increase smoothly, with a maximum value of only 743 gammas (a value not easily recognized as unreasonable). There is some oscillation on the sides of the central peak, but even these are minor. In contrast to this result, the spectral method yielded maps with noticeable oscillations at levels about one mile above the source.

The field on the axis of a long vertical cylinder can be approximated by the field of a magnetized disc given as

$$\Delta H(Z)=2\pi kH_v\left[1-\frac{Z}{\sqrt{a^2+Z^2}}\right], \qquad (9)$$

where k is the magnetic susceptibility, Hv is the earth's vertical magnetic field and a is the radius of the disc. The first and second derivatives are

$$\frac{\partial(\Delta H)}{\partial Z}=2\pi kH_v\left[\frac{a^2}{(a^2+Z^2)^{3/2}}\right]. \qquad (10)$$

and

$$\frac{\partial^2(\Delta H)}{\partial Z^2}=6\pi kH_v\left[\frac{a^2Z}{(a^2+Z^2)^{5/2}}\right]. \qquad (11)$$

Using the Henderson and spectral method, first and second derivatives were computed for the model cylinder at the surface and at a level of 0.5 mile below the surface (Figs. 7 and 8). First and second derivative calculations on the surface yield similar curves as displayed on the cross-sections. The "true" values over the axis of the cylinder were calculated from equations (10) and (11) and are close to that obtained from the computer output (these results are also given in table 2).

The derivative calculations on a level 0.5 mile below the surface are also plotted on figures 7 and 8. In general, both methods yield smooth functions in the central part of the anomaly, but the spectral method developes strong oscillations on the flanks. However, the peak value from the spectral method is very close to the true value, while the Henderson method is in error. This result is again consistent with the view that the spectral results may be more reliable over the central portion of the anomaly.

5. Conclusions

For upward continuation, Henderson's method and the spectral method are equally applicable and yield similar results. For downward continuation and derivative calculation some discrepancies are observed between the two methods. Our results show the spectral method computes near correct values over the central part of the anomaly. At the edge of the anomaly, the spectral method developes serious oscillations 1.2 miles below the surface (the body is at a depth of two miles). These oscillations gradually extend toward the

Table 2. True and computed values of the first and second derivatives of the magnetic field on the axis of a vertical cylinder (see Fig. 1). Computed values from methods of Henderson (1960) and Argawal (1968): true values from Equation (10, 11).

first derivative, $\frac{\partial(\Delta\phi)}{\partial Z}$, gammas/miles				second derivative, $\frac{\partial^2(\Delta\phi)}{\partial Z^2}$, gammas/miles^2			
depth z	true values	comp±ued values and % error		depth Z	true values	computed values and % error	
		Henderson	spectral			Henderson	spectral
0	62	58 (-6%)	59 (-5%)	0	74	64 (-13%)	73 (-1%)
1	118	98 (-17%)	114 (-3%)	1	164	108 (-34%)	155 (-5%)
2	245	170 (-30%)	*	2	367	176 (-52%)	*
3	496	280 (-43%)	*	3	595	272 (-54%)	*

*Oscillatory values —not considered meaningful.

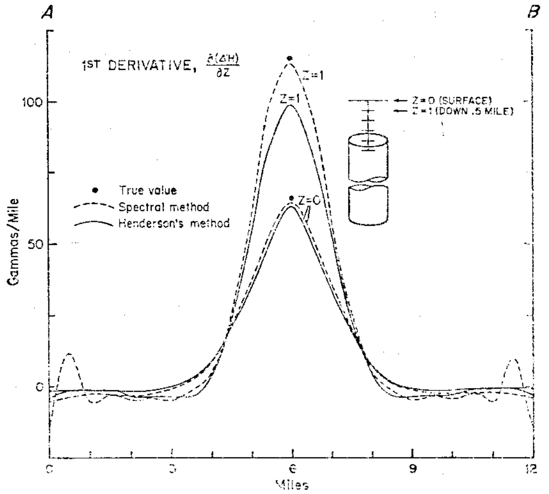

Fig. 7 Cross section of the first derivatives of the magnetic field at two levels. Values computed by Henderson's method and spectral method. True values over the axis of the cylinder computed from equation (10).

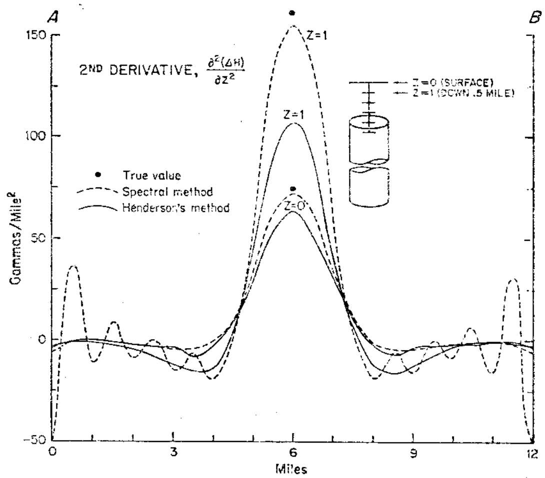

Fig. 8 Cross section of the second derivatives of the magnetic field at two levels. Values computed by Henderson's method and spectral method. True values over the axis of the cylinder computed from equation (11).

center of the map as we continue downward or increase the order of derivatives.

It is assumed that spectral values near the edge of the map are intrinsically unstable. Because Fourier transformation assumes an infinite number of data points or a perfect periodic function, limited input data introduces a discontinuity at the edge of the map. Battacharya (personal communication) has suggested that a recursive filter can yield maps of significant improvement. This suggests a new area of future research. Although not presented in this study, the gravity field of our model cylinder was also investigated. However, the potential gradient between the center and edge of the map is small and the increased discontinuity yields more severe oscillations than those observed from the magnetic field.

The seriousness of the discrepancies is illustrated by comparing values over the axis of the cylinder. The computed values for continued fields differ from the true value less than three gammas for all upward continued fields (Table 1). For downward continued fields, the spectral method yields good results at a depth of one mile (2 grid units) and then blows up. The Henderson's method is still meaningful at 1.5 miles. We suggest, however, that the reader use caution in applying values continued downward to levels more than three-fourths of the depth to the body. Values should oscillate in the proximity of the body, but Henderson's methods seems to subdue these effects in the case of the model.

Discrepancies in first and second derivatives are also summarized by comparing the values over the axis of the cylinder (Table 2). The spectral method again yields values close to that predicted analytically (errors of only 5%). However, the derivative on a level 1.0 mile down blows up. Henderson's method does not severely oscillate as predicated.

Our results suggest that the spectral approach may be improved by extending the map size and thereby avoiding the edge effects. In the case of real (field) data, the values should be carefully smoothed with a high frequency filter. The sudden oscillations that appear all over the map is theoretically consistent with Roy's observation (1967) that the maximum level of convergence is a horizontal plane at the top of the body. Henderson's method seemingly yields smooth values for fields continued quite close to the body. The lack of prominent oscillations indicates that continuation must be applied cautiously as a valid indicator of body depth. However, negative values appear at levels close to the body and this may be sufficient to identify the top of the body. No theoretical calculations have been made to predict the shape of the convergence surface for Henderson's method. Future studies along these lines may be necessary to explain the distribution of oscillations.

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포텐셜場의 上下向連續 및 微分法에 對한 比較研究

權 炳 杜

요약 : 磁場 또는 重力場의 上向 및 下向連續은 適用方法에 따라 差異를 보임이 模型研究에 의해 밝혀졌다. 垂直圓柱型의 모델로부터 計算한 磁力場을 Henderson의 方法(Lagrange의 內挿法應用)과 스펙트럼方法(周波數領域에서의 解析法)을 利用하여 여러 深度로 上向 및 下向連續値를 計算한 바 다음과 같은 結論을 얻었다. (1) 上向連續値는 別다른 差異가 없다. (2) 下向連續은 異常帶의 中央部에서는 스펙트럼方法으로 計算한 값이 더 正確하였으며, (3) 異常帶의 가장자리에서는 Henderson의 方法에 의한 計算値가 理論値에 더욱 가깝다. 스펙트럼方法으로 計算한 값은 鑛體에 가까운 深度에서는 oscillation을 나타내며, 微分法을 適用한 結果도 以上과 대체로 類似하다.

스펙트럼方法에 의한 下向連續値는 鑛體深度의 1/2程度되는 深度에서부터 oscillation하는 값들을 보이기 시작하나, Henderson의 方法을 適用하였을 때는 鑛體直上部에서도 가장자리에 약간의 oscillation을 보일뿐이며 鑛體를 지나 계속 下向連續을 試圖하였을 때도 완곡 異常曲線을 보인다. 따라서 下向連續의 計算結果에서 나타나는 oscillation으로부터 鑛體의 深度를 推定할 때는 어떤 方法을 適用하였는가를 有意해야 한다. 마찬가지로 微分計算値로부터 鑛體의 輪廓을 決定할 때도 計算方法을 고려해야 한다.