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The Relation between $H^0(X)$ and Hom $(H_0(X), Z)$

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§ 1. Introduction

In this paper we will investigate an algebraic strueture of $H^0(x)$ and $H_0(X)$.

Let $\pi_0(x)$ be the set of equivalent classes of points of X under the relation which there exits a path in X joining x to y for any two points $x, y \in X$. and let $F(\pi_0(x))$ be the free abelian group generated by $\pi_0(X)$, then we will prove that there exists an injective homomorphism between $H^0(X)$ and Hom $(H_0(X), Z)$.

§ 2. Definitions and Preliminaries

Definition 1. Let X be a topological space,

$H^0(X) = \{f \mid f: X \to Z$ is continuous$\}$,

where Z is integers.

Lemma 1.

$H^0(X)$ forms an abelian group.

Proof. It is obvious.

Definition 2.

Let X and Y be topological spaces and $f: X \to Y$ a cantinuous map. Define $f^*: H^0(Y) \to H^0(X)$ by $f^*(g) = g \cdot f$.

Lemma 2.

f^* is a homomorphism. If 1 is the identity map of X, 1^* is the identity of $H^0(X)$.

If $f: X \to Y$ and $g: Y \to W$, then $(g \cdot f)^* = f^* \cdot g^*$.

Proof.

$f^*(g_1 + g_2)(x) = (g_1 + g_2)(f(x)) = g_1(f(x)) + g_2(f(x))$

$= f^* g_1(x) + f^* g_2(x) = (f^* g_1 + f^* g_2)(x)$,

so we have a homomorphism.

The second statement is clear, so is the third on account of

$(g\circ f)^* h = h\circ(g\circ f) = (h\circ g)\ \text{of} = f^*(h\circ g) = (f^*\circ g^*)h.$

Lemma 3.

Suppose that $\pi_0(X)$ is the set of equivalent classes of points of X under the relation which there exists a path in X joining x to y for any two points in X and $f: X\to Y$ a continous map, then f induces a map

$f_*: \pi_0(X)\to\pi_0(Y).$

If f is the identity map of X, f_* is the identity map of $\pi_0(X)$,

If $f: X\to Y$ and $g: Y\to W$ then $(g\circ f)_* = g_* f_*$.

Proof.

The way of proof of this is similar to (Lemma 2).

Proposition 1.

Let $f: X\to \mathbf{Z}$ belong to $H^0(X)$. Then, if $x\sim x'$ as points of X,

$f(x) = f(x')$. Thus f factorizes as a function on the set of equivalence classes,

$$X \to \pi_0(X) \xrightarrow{c(f)} \mathbf{Z}.$$

Denote map $(\pi_0(X), \mathbf{Z})$ for the set of all integer—valued functions on the set $\pi_0(X)$.

Then pointwise addition of functions gives Map $(\pi_0(X), \mathbf{Z})$ the structure of an abelian group.

The map defined above,

$$C: H^0(X) \to \text{Map}(\pi_0(X), \mathbf{Z})$$

is an injective Homomorphism of abelian groups.

Proof.

If $x\sim x'$, we can find a path $P: I\to X$ joining x to x'.

Then $f\circ p: I\to \mathbf{Z}$ is continuous.,

it is constant.

Thus $f(P(0)) = f(P(1))$, that is, $f(x) = f(x')$. Indeed, we can alternatively argue that the relation $\sim$ is trivial on Z,

so that $\pi_0(\mathbf{Z}) = \mathbf{Z}$.

Then $c(f) = \pi_0(f)$.

It is clear that Map $(\pi_0(X), \mathbf{Z})$ is an abelian group.

Let β be the equivalence class of a general point $x\in X$.

That c is a homomorphism follows from the computation,

$c(f_1+f_2)(\beta)=(f_1+f_2)(x)=f_1(x)+f_2(x)$

$=c(f_1)(\beta)+c(f_2)(\beta)=(c(f_1)+c(f_2))(\beta)$,

implying $c(f_1+f_2)=c(f_1)+c(f_2)$.

Finally, if $P: X \to \pi_0(X)$ denotes the projection, then by definition of c,

$f=c(f)\circ p$. Clearly, then, if $c(f)=c(g)$,

We have $f=c(f)\circ p=c(g)\circ p=g$.

Proposition 2.

Let X be locally path-connected(l. p. c).

Then its path-components are open in X.

Proof.

Let β be a path-component, $x\in\beta$.

Since X is *l. p. c.* at x, x has a path-connected neighborhood U.

All points of U are joinable to x, So $U\subset\beta$.

As β contain a neighborhood of each of its points, it is open

Proposition 3.

If X is *l. p. c.*,

then

$c: H^0(X) \to \mathrm{Map}(\pi_0(X), \boldsymbol{Z})$ is an ismorphism.

Proof.

We already know that c is an injective homomorphism, it remains to show, then, that c is surjective.

Now c was characterized by $f=c(f)\circ p$.

Thus a map $F: \pi_0(X) \to \boldsymbol{Z}$ is in the image of c if and only if

$f=F\circ p: X \to \boldsymbol{Z}$ is continuous.

Now for each $M\in\boldsymbol{Z}$, $f^{-1}\{n\}$ is the union of the path-components, β such that $F(\beta)=n$. By (proposition 2),

Since X is *l. p. c.*, these are open, hence so, is their union.

Thus f is continuous.

Definition 3.

$H_0(X)=F(\pi_0(X))$, where $F(\pi_0(X))$ is the free abelian group over $\pi_0(X)$.

§ 3. Main theorem

Theorem

For any X, $(i^*)^{-1} \circ c = K : H^0(X) \to \mathrm{Hom}(H_0(X), \mathbf{Z})$ is an injective homomorphism of abelian groups.

If X is *l. p. c.*, it is an isomorphism.

Proof.

The projection $P : X \to \pi_0(X)$ induces a group homomorphism

$F(P) : F(X) \longrightarrow F(\pi_0(X)) = H_0(X)$,

which is characterized by requiring the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad P \quad} & \pi_0(X) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ F(X) & \xrightarrow[\quad P^* \quad]{} & F(\pi_0(X)) \end{array}$$

to commute.

From the universal mapping property of $F(\pi_0(X))$, we deduce that,

$i^* : \mathrm{Hom}(H_0(X), \mathbf{Z}) \to \mathrm{Map}(\pi_0(X), \mathbf{Z})$ is an isomorphism.

If we combine this with (proposition 1) and (proposition 3), we get the main theorem.

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