HONAM MATHEMATICAL JOURNAL
Volume I, Number 3, Sep. 1981

The Greek formal logic and the axiomatic method in stoicheia

Yong Kook Kim

§ 0. Introduction

Originally, logic was closely related to mathematical speculation and one can see one of modern illustrations in Russell / whitehead's Principia Mathematica. Besides, in many mathematics text books of Bourbaki's style, there appears to be no essential distinction between these two. In fact, there is a view that goes so far as to define explicitly that (modern) logic equals (axiomatic) set theory.

This thesis seeks a possible relationship between the axiomatic method in Stoicheia and formal logic. It also aims at inquiring into the functions of logic related to the axiomatic system of mathematics. It is considered advisable, therefore, in this study to exclude all the outer fields of mathematics proper, such as social, linguistic and other conditions of the ancient Greece, as well as possible influences exerted on it by the Orient mathematics. This has led us to decide not to refer to A. Szabo's study on the terminology in Stoicheia, connected with that of Greek philosophy before it, though it would certainly draw our interest.①

Logical symbols used in this paper generally concur with those in Principia Mathematica.

§ 1. The Greek formal logic before Stoicheia

1) Before Aristotle

Though no system of formal logic can be traced in Greece before Aristotle's Topica, several of its rules are seen to have been used consciously among philosophers since Eleatics. Parmenides (544~501 B. C), in criticizing Milesians' Physica of Becoming', employs the next logical propositions:

① A. Szabó, "*Anfänge der griechischen Mathematik*"

$$A_x \supset (B_x . \sim B_x) . \supset . \sim A_x$$

According to Simplikios,[1] Zenon, Parmenides' succesor, used what falls under the following :

$$A_x . \supset . B_x . C_x : \sim (B_x . C_x) : \supset : \sim A_x$$

It seems obvious that these logical laws are derived from dialectics, a fundamental rule for indirect proof employed in countering the opponents' assertion at a forum. As is well known, indirect proof was frequently used under the name of reductio ad absurdum (η' εἰδτο' αδυνατν απαγωγη) in these days.

The following type of syllogism by Zenon is also noted :

$$A_x \supset B_x . B_x \supset C_x . \supset . A_x \supset C_x$$ [2]

Plato is said to have adopted the following in one of his writings, Who he did attempt to construct a formial logic :

$$A_x \supset \sim A_x . \supset . \sim A_x$$ [3]

Gorgias (483~376 B. C), a representative of sophists, presumably used the contradiction mentioned below :

$$(x) . \sim (A_x . \sim A_x)$$ [4]

2) Formal logic of Aristotle

Aristotle excludes the following inductions as invalid, in explining 'the fallacy of conclusion' (παρὰ τὸ ἑπόμενον), related to 'Sophistics', Topcia.

$$p \supset q . q . \supset . p$$ [5]

$$p \supset q . \sim p . \supset . \sim q$$

His Laws of Contradiction and of the Excluded Middle can be symbolized as follows :

$$(x . \varphi) . \sim (\varphi x . \sim \varphi x)$$ [6]

---

① Simpl. Phys. 140, 34; D 1, 255.
② ibid 140, 27 ff; D 3, 257 f.
③ Theat. 171 a.
④ J. M. Bochénski, "*Ancient formal logical*" 1968, 9. 24.
⑤ Soph. E 1. 28, 181a 27 ff. 167b
⑥ Met. Γ 3, 1005b 19f.

$(T^{\ulcorner}\varphi x^{\urcorner} \cdot T^{\ulcorner}\sim\varphi x^{\urcorner}$ [1]

$(x, \varphi).\ \varphi x V \sim \varphi x$ [2]

$T^{\ulcorner}\varphi x^{\urcorner}\ V T^{\ulcorner}\sim\varphi x^{\urcorner}.\ \ F^{\ulcorner}\varphi x^{\urcorner}\ V F^{\ulcorner}\sim\varphi x^{\urcorner}$ [3]

$T^{\ulcorner}\varphi x^{\urcorner}\ \ V F^{\ulcorner}\varphi x^{\urcorner}$ [4]

Categorical syllogism can be symbolized as :

$B_x \supset (x)\ \ A_x$ [5]

$(x)\, B_x \supset (x)\ \ A_x$

Aristotle refers emphatically to axiomatization that firstly there must be several unprovable axioms (ἀξιώματα) in order to deduce other theorems out of them[6]; secondly the number of operations in each system of axioms needs to be finite[7]; and thirdly logical axioms should be evident by intuition.[8] In fact, he applies this reasoning to the system of syllogism, he regards a few logical propositions as basic axioms, from which others are to be deduced.

The following four are assumed as the fundamental axioms :

MaP. SaM. ⊃ . SaP [9]

MeP. SaM. ⊃ . SeP [10]

MaP. SiM . ⊃ . SiP [11]

MeP. SiM . ⊃ . SoP [12]

Based on the above axioms the following are deduced :

Ⅰ. PeM . SaM ⊃. SeP [13]

Ⅱ. PaM · SeM ⊃. SeP [14]

Ⅲ. PeM · SiM ⊃. SoP [15]

Ⅳ. PaM · SoM ⊃. SeP [16]

Ⅴ. MaP · MaS ⊃. SiP [17]

① ibid 6, 10116 15f. This proposition can also be expressed as follows :
$\sim(T^{\ulcorner}SaP^{\urcorner}.\ T^{\ulcorner}SoP^{\urcorner}).\ \sim(T^{\ulcorner}SeP^{\urcorner}.\ T^{\ulcorner}SiP^{\urcorner})$
where SaP means '$\forall s,\ P \in S$. SoP, '$\exists s,\ D \notin S$', Sep, '$\forall S,\ P \notin s$', Sip, $\exists S,\ P \in S$'
②~③ idid ④ De Interpretatione, 9, 18a 37f ⑤ Prior Analytics A41, 49b 14ff.
⑥ Posteriot Analytics, A3, 72b 18 f. ⑦ ibid A19~20, 81b ⑧ ibid B19, 99b 20ff
⑨De Int. 10, 20a 20f, It is called 'Barbara'. ⑩ ibid, It's Celarent' ⑪ ibid, Darii
⑫ ibid, Feris, As Aristotle himself recognizes later,
however, the last two propositions ③and ④ are unnecessary, the first two implying them

⑬~⑰De Int 10, 20*a* 20*f* *A* 4, 5, 6

Ⅵ. MeP. MaS . ⊃. SoP ①

Ⅶ. MiP. MaS. ⊃. SiP②

Ⅷ. MaP. MiS . ⊃. SiP ③

Ⅸ. MoP. MaS. ⊃. SoP ④

Ⅹ. Mep. MiS . ⊃. SoP ⑤

Regarding the logic of sets and the Predicate, there are :

$A \subset B . \supset . -B \subset -A$ ⑥

$-A \subset -B . \supset . B \subset A$ ⑦

$(x) A_x \supset (\exists_x) A_x$ ⑧

$(x) \sim A_x \supset \sim (x) A_x$ ⑨

$(x) A_x \supset \sim (\exists_x) \sim A_x$ ⑩

$(x) A_x \supset \sim [(\exists_x) A_x , (\exists_x) \sim A_x]$ ① ⑪

$(x) \sim A_x \supset \sim [(\exists_x) A_x . (\exists_x) \sim A_x]$ ⑫

Laws of hypothetical Syllogism are :

Ⅰ. $A_x \supset B_x . A_x . \supset . B_x$ ⑬

Ⅱ. $A_x \supset B_x . B_x \supset C_x . \supset . A_x \supset C_x$ ⑭

Ⅲ. $A_x V B_x . A_x . \supset . \sim B_x$ ①

Ⅳ. $A_x V B_x . B_x . \supset . \sim A_x$ ②

Ⅴ. $A_x V B_x . \sim A_x . \supset . B_x$ ③

Ⅵ. $A_x V B_x . \sim B_x , \supset . A_x$ ④

Ⅶ. $(x). \sim (A_x . B_x) : (x). A_x \vee B_x : (x). \sim (C_x . D_x) : (x).$
$C_x \vee D_x : (x). C_x \supset A_x : \supset : (x). B_x \supset D_x$ ⑤

Ⅷ. $(x). \sim (A_x . B_x) : (x) . A_x \vee B_x : (x). \sim (C_x . D_x) : (x).$
$C_x \vee D_x : (x). C_x \supset A_x : \supset : (x). \sim (B_x . C_x)$ ⑥

① ~⑤ De Int. 10, 20a 20f, A 4, 5, 6The logical expressions from I to X can be summerized as the following three.

MaP . SaM . ⊃. Pis (Baralipton)
MaP . SiM .⊃. PiS (Dabitus)
MeP . SaM . ⊃. PeS (Celantis)

⑥ ~ ⑫ ToP. B 8, 113 b 17f, ibid. B 2

⑬ An. Pr. A44. 50a 19ff. This can be substituted for the following.

$\sim A_x \supset \sim B_x . \sim A_x . \supset . B_x$

⑭ ibid A32, 47a 28ff.

①~④ Top B 5, 112a 24 – 30, 'V' (capital of 'V') seen from Ⅲ to Ⅵ means exclusive disjunction.

⑤An. Pr. A46, 52a 39ff ⑥This proposition can be simplified as follows :

$(x). A_x \vee B_x : (x). C_x \vee D_x : (x). C_x \supset A_x$

In regard to identity theory, the following are shown :

$x=z,\ y\neq z.\supset.\ x\neq y$ ①

$x=y.\supset.(A).A_x\supset A_y$ ②

$A=B.\supset.(x).A_x\supset B_x$ ③

$\sim(A).A_x\supset A_y.\supset.\ x\neq y$ ④

$\sim(x).A_x\supset B_x.\supset.\ A\neq B$ ⑤

$x\neq y\ \supset.\sim(A).A_x\supset A_y$ ⑥

From the relation hitherto dealt with, Aristotle deduces, in conclusion, the following propositional calculi :

$p\supset q.\supset.\sim q\supset\sim p$ ⑦

$p\supset q.\ T\ulcorner p\urcorner.\supset.T\ulcorner q\urcorner$ ⑧

$p_1p_2\cdots p_n.\supset.\ r:\supset: F\ulcorner r\urcorner.\supset.F\ulcorner p_1\urcorner\vee\cdots\vee F\ulcorner p_n\urcorner$ ⑨

§2. The axiomatic method and formal logic in Stoicheia

In A Commentary on Stoicheia Vol. I Proklos said;

"Eucleides had faith in Platonism, having a good knowledge of the philosophy, and wrote Stoicheia for the ultimate purpose of constructing what is called Platonic Figure."

From the above quotation, we can infer that Eucleides was considerably interested in the logic of Plato and Aristotle.

K. V. Fritz mentions repeatedly in his paper⑩ that Aristotle intended to verify that every systematic knowledge should start from several unprovable, yet true and firm principles. Indeed, it can be said that Aristotle did prepare for building up the foundation of an axiomatic mathematics. For it was from this recognition of unprovability governing these principles that Greek mathematics has developed its definitions, postulates and axioms. In other words, Greek mathematics is built on what was accomplished by Aristotle himself.

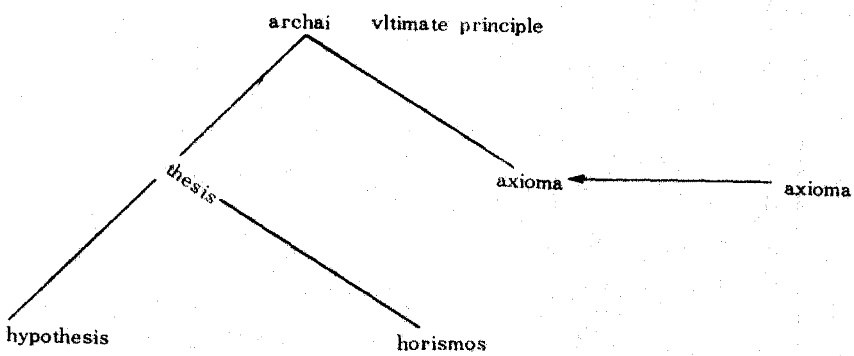
Aristotelian diagram for classfying them may be as follows. ⑪

①~⑤ibid

⑥ Soph. E 1. 24. 179a 37~39. ⑦~⑨ An. P. B_2 53b 12ff.

⑩ K. V. Fritz, " Die 'ἀρχαι in der *griechischen mathematik*" S. 64~65.

⑪ A. Szabo, ibid S. 275.



Stoicheia (Στοιχεῖα) originally means 'Alphabet'. This implies that all the propositions on mathematics (=geometry) should be proved and systematized according to stoicheia, as each language is founded on alphabet. In composing stoicheia, it is most important to prove propositions, which pile up to organize an axiomatic system.

Logic used in Stoicheia is very strict, and covers completely all the range of what is called predicate logic today. From the very beginning, in proposition 1 of vol. I (that is, "to draw an equilateral triangle on a given finite straight line (=segment)"), a method of syllogism is used :

$(x).\ (y).\ (z).\ A(x, y, z).\ \supset.\ B(x, y, z) : (\exists a).(\exists b).(\exists c).$
$A(a, b, c) :\supset: B(a, b)$

It is worth noting closely the fact that, among deductions composing the axiomatic system of Stoicheia, there are frequently used indirect methods of proof, reductio ad absurdum, in addition to direct ones, especilly in Vols VII, VIII and IX. That is :

Vol. VII : prop. 1, 2, 3, 20, 21, 22, 23, 24, 28, 29, 33, 34, 35, 39, (36% of the total)

Vol. VIII : prop. 1, 4, 6, 7, 16, 17, (22%)

Vol. IX : prop. 10, 12, 13, 14, 16, 17, 18, 19, 20, 30, 31, 33, 34 (36%)

In general, reductio ad absurdum as a method of proof comprises the following three steps :

① If '$\sim P$' is to be assumed,

② Then, it comes to contradiction

(In a definite shape, it can be expressed that a proposition Q which is inevit -

ably lead out of '$\sim P$' is in compatible with the premise having been proved or assumed in advance. Namely, this result reaches against the law of contradiction.)

③ Therefore, it leads in conclusion to 'P'.

To sum up the above process, we obtain an expression of propositional logic.

$$\sim P. \supset. (Q, \sim Q) : \supset : P$$

On the other hand,

$$F^{\ulcorner}\sim P^{\urcorner} \supset T^{\ulcorner}P^{\urcorner}$$

$$i.\ e.\ \sim (T^{\ulcorner}\varphi(x)^{\urcorner} \quad {}^{\ulcorner}T\sim\varphi(x)^{\urcorner})$$

is on the assumption of the excluded middle; that is

$$P \vee \sim P$$

$$i.\ e.\ T^{\ulcorner}\varphi(x)^{\urcorner} \vee F^{\ulcorner}\varphi(x)^{\urcorner}$$

In short, reductio ad absurdum contains in itself the law of contradiction and of the excluded middle.

It is well known that the formative period of Vols. Ⅶ～Ⅸ precedes that of other vols. Szabó has made clear in his writing① that Greek mathematicians about 5 B. C— who were the followers of Pythagoras-introduced the method of reasoning of Eleatics, together with an anti-empirical and anti-explanatory approach, by which they developed a deductive method in mathematics and succeeded in shapng theoretical mathematics.

§ 3. Conclusion

As we see clearly in Aristotles' Metaphysics, he was avidly interested in mathematics of his days, and also fully recognized that his logical laws were rather defective in explaining all the logical operations by mathematicians.

Mathematics borrowed profusely from philosophy in terms of reasonings, but has altered and refined the modes of reasoning to suit its convenience and particular purposes, thus making them unique. This is why a mathematician, without prior knowledge of logic, easily— and perfectly— demonstrates propositions. However, one should be cautioned against jumping rashly from mathematics to philsophy or vice versa.

This presents a delicate problem regarding the relationship between mathematics and philosophy in the field of the foundation of mathematics.

However, it will be in order to put forward a bold question : Can mathematics

① ibid 3. 246.

as an axiomatic system be shaped solely and independently disregarding complet-ely logic ? In our hypothesis, the answer to this is negative. this is suggested becau se we are aware in the first place that, in modern mathematical history, symbolic logic including G. Bools logical algebra emerged around the time when mathema-tics began to proceed towards axiomatization. In the second place, there has been no trace of axiomatic system in the Chinese history of mathematics, though fragmen tary records are available on axiomatic geometry during the turbulent ancient time (770～221 B. C) when many logicians were highly active.

References

1. Aristoteles, *Prior and Posterior Analytics*, ed. W. D. Ross Oxford 1949.
2. Aristoteles, *Metaphysics*, ed. W. D. Ross Oxford 1948.
3. Bocheński, J. M. *Ancient formal logic*, 1968.
4. Cohlke, D. *Die Entstehlung der Aristotelischen Logik*, Berlin, 1936.
5. Fritz, K. V. *Die ἀρχαί in der griechischen Mathematik*, 1945.
6. Hurst, M. *Implication in the 4 th century B. C.*, Mind 44, 1935,
7. Kapp, E. *Greek Foundations of Traditional Logic*, New York, 1942.
8. *Stoicheia* (Thomas L. Heath, *The thirteen books of Euclid's Elements*, translated from the text of Heiberg. 3 vols, New York 1956).

9. Szabó, A. *Anfänge des euklidischen Axiomen systems*, 1960.
10. Szabó, A. *Anfänage der griechischen Mathematik*, 1969.

(Mokpo College)