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ON ALLOCATION IN STRATIFIED SAMPLING BASED ON PRELIMINARY TEST OF SIGNIFICANCE.

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1. Introduction.

The Neyman allocation of stratification, one type of sampling method, depends upon strata variances which are generally not known.

One way to overcome this difficulty is to use the technique of two-phase sampling introduced by Sukhatme [7]. This technique consists in first drawing a preliminary sample of fixed size from each stratum to estimate σ_i^2. This allocation is called the modified Neyman allocation.

If the strata variances σ_i^2 differ significantly among themselves, Evans verified that the modified Neyman allocation is more efficient than the proportional allocation. But if the strata variances σ_i^2 do not differ significantly among themselves, the modified Neyman allocation turned out to be less efficient than the propotional allocation [4].

In [2] the efficiency of the proportional- Neyman allocation in case of $\sigma_1^2 \leq \sigma_2^2$, is in investigated in which proportional allocation is used if the proportion of two strata variances $s_j^2/s_i^2<\lambda$ and modified Neyman allocation is used if $s_j^2/s_i^2 \geq \lambda$, λ is a fixed constant. But in this paper, the efficiency of proportional-Neyman allocation is compared with the efficiency of proportional allocation in the case of $\sigma_1^2 \neq \sigma_2^2$.

2. Variance of $\bar{y}_w$, the population mean estimate.

If two population variances differ among themselves, namely $\sigma_1^2 \neq \sigma_2^2$, the proportional-Neyman allocation takes the form.

$$n_i = nw_i \ (s_j^2/s_i^2<\lambda \ \text{for} \ i\neq i)$$
$$= nw_i s_i/\sum w_i s_i \ (s_j^2/s_i^2 \geqq \lambda \ \text{for} \ i\neq i)$$

The proportional allocation is used if $s_j^2/s_i^2<\lambda$, and the Neyman allocation is used if $s_j^2/s_i^2 \geq \lambda$. Let the event B_0 be defined by B_0 : { $s_j^2/s_i^2<\lambda$ for $i, j=1,$

2. $i \neq j$] and B_1 be the complementary event of B_0. Following [2], the variance of the estimate $\bar{y}_w$ under the proportional-Neyman allocation is given by

$$V(\bar{y}_w)_{P.N} = \sum_{i=0}^{1} E_i[V(\bar{y}_w \mid B_i)]\ P(B_i) \tag{2.1}$$

where E_i denotes that the expectation is taken with reference to the set B_i.

Again using [5],

$$E_0[V(\bar{y}_w \mid B_0)] = V(\bar{y}_w)_p = \left(\frac{1}{n}-\frac{1}{N}\right)\sigma_1^2\ (W_1 + W_2\ \theta_{21})$$

$$E_1[V(\bar{y}_w \mid B_1)] = \frac{1}{n}\sum_{i=1}^{2} W_i^2\ \sigma_i^2 - \frac{1}{N}\sum_{i=1}^{2} W_i\sigma_i^2 + \frac{1}{n}\sum_{(\neq)}^{2} W_i W_j \sigma_i^2 E[s_j/s_i \mid B_1] \tag{2.2}$$

To eavaluate $E[s_j/s_i \mid B_1]$, We use another result due to Carrillo [1] given below.

Lemma 1. Let S_1^2 and S_2^2 be independent unbiased estimates of σ_1^2 and σ_2^2 based on f_1 and f_2 degrees of freedom respectively.

Then using [1], we obtain

$$E[S_1^{2t_1}S_2^{2t_2} \mid B_1]P[B_1] = [Ip_{21}^{(1)}\ (\tfrac{1}{2}f_1+t_1,\ \tfrac{1}{2}f_2+t_2) - Ip_{21}^{(2)}(\tfrac{1}{2}f_1+t_1,\ \tfrac{1}{2}f_2+t_2)] \cdot \sum_{i=0}^{2}(2\sigma_i^2/f_i)^{ti}\,\Gamma(\tfrac{1}{2}f_i+t_i)/\Gamma(\tfrac{1}{2}f_i) \tag{2.3}$$

$$E[S_1^{2t_1}S_2^{2t_2} \mid B_1]\ P(B_1) = [I_{q21}(1)\ (\tfrac{1}{2}f_2+t_2,\ \tfrac{1}{2}f_1+t_1) + Ip_{21}^{(2)}\ (\tfrac{1}{2}f_1+t_1,\ \tfrac{1}{2}f_2+t_2)]\ .\ \prod_{i=1}^{2}(2\sigma_i^2/f_i)^{t_i}\ \Gamma(\tfrac{1}{2}f_i+t_i)/\Gamma(\tfrac{1}{2}f_i) \tag{2.4}$$

where $P_{21}^{(1)} = \dfrac{1}{1+(f_2\sigma_1^2\lambda_{21}^{(1)}/f_1\sigma_2^2)}$ and $q_{21}^{(1)} = 1 - P_{21}^{(1)}$ Letting $t_1 = -\frac{1}{2}$, $t_2 = \frac{1}{2}$, $f_1 = f_2 = f$, $\lambda_{21}^{(1)} = \frac{1}{\lambda}$, $\lambda_{21}^{(2)} = \lambda$

in (2.4). We obtain

$$E[S_1/S_2 \mid B_1]\ P(B_1) = \frac{1}{2}G\ \theta_{21}^{\frac{1}{2}}\ [I_{q21}^{(1)}\ (\tfrac{1}{2}f+\tfrac{1}{2},\ \tfrac{1}{2}f-\tfrac{1}{2}) + Ip_{21}(\tfrac{1}{2}f-\tfrac{1}{2},\ \tfrac{1}{2}f+\tfrac{1}{2})] \tag{2.5}$$

where $q_{21}^{(1)} = 1 - P_{21}^{(1)} = \dfrac{1}{1+\lambda\theta_{21}}$, $G = 2\,\Gamma(\tfrac{1}{2}f-\tfrac{1}{2})\Gamma(\tfrac{1}{2}f+\tfrac{1}{2})/\Gamma(\tfrac{1}{2}f)\Gamma(\tfrac{1}{2}f)$.
and similarly,

$$E[S_1/S_2 \mid B_1]\ P(B_1) = \frac{1}{2}G\theta_{21}^{-\frac{1}{2}}\ [Iq_{21}^{(1)}\ (\tfrac{1}{2}f-\tfrac{1}{2},\ \tfrac{1}{2}f+\tfrac{1}{2}) + Ip_{21}\ (\tfrac{1}{2}f+\tfrac{1}{2},\ \tfrac{1}{2}f-\tfrac{1}{2}) \tag{2.6}$$

To evaluate $P(B_0)$, let $t_1 = t_2 = 0$ in (2.3). The result is

$$P(B_0) = Ip_{21}^{(1)}(\frac{1}{2}f, \frac{1}{2}f) - Ip_{21}(\frac{1}{2}f, \frac{1}{2}f) \qquad (2.7)$$

$$P(B_1) = 1 - P(B_0) = Iq_{21}^{(1)}(\frac{1}{2}f, \frac{1}{2}f) - Ip_{21}(\frac{1}{2}f, \frac{1}{2}f) \qquad (2.8)$$

Using (2.5) (2.6) and (2.8), we obtain

$$E_1[V(\bar{y}_w \mid B_1)] = \sigma_1^2(w_1^2 + w_2^2\theta_{21})/n - \sigma_1^2(w_1 + w_2\theta_{21})/N + \frac{1}{2}G\frac{w_1 w_2}{n}\sigma_1^2\theta_{21}^{\frac{1}{2}}$$
$$\cdot [Iq_{21}^{(1)}(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2}) + Ip_{21}(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f + \frac{1}{2}) + Iq_{21}^{(1)}(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f$$
$$+ \frac{1}{2}] + Ip_{21}(\frac{1}{2}f + \frac{1}{2}, \frac{1}{2}f - \frac{1}{2})] / [Iq_{21}^{(1)}(\frac{1}{2}f, \frac{1}{2}f) + Ip_{21}(\frac{1}{2}f, \frac{1}{2}f)$$
$$(2.9)$$

Substituting (2.2) (2.7) (2.8) and (2.9) in (2.1), we obtain

$$V(\bar{y}_w)_{P.N} = \sigma_1^2(w_1^2 + w_2^2\theta_{21})/n - \sigma_1^2(w_1 + w_2\theta_{21})/N + \frac{w_1 + w_2}{n}\sigma_1^2\{(1 + \theta_{21})[Ip_{21}^{(1)}(\frac{1}{2}f,$$
$$\frac{1}{2}f) - Ip_{21}(\frac{1}{2}f, \frac{1}{2}f)] + G\theta_{21}^{\frac{1}{2}}[Iq_{21}^{(1)}(\frac{1}{2}f - \frac{1}{2}, \frac{1}{2}f - \frac{1}{2}) + Ip_{21}(\frac{1}{2}f - \frac{1}{2},$$
$$\frac{1}{2}f - \frac{1}{2})]\}.$$

If we let λ tend to infinity, we obtain the variance of the estimate $\bar{y}_w$ under proportional allocation. Putting $\lambda = 1$, we obtain the variance of $\bar{y}_w$ under the modified Neyman allocation.

Hence we obtain

$$V(\bar{y}_w)_P = (\frac{1}{n} - \frac{1}{N})\sigma_1^2(w_1 + w_2\theta_{21})$$

$$V(\bar{y}_w)_N = \frac{\sigma_1^2}{n}(w_1^2 + w_2^2\theta_{21}) - \frac{\sigma_1^2}{N}(w_1 + w_2\theta_{21}) + \frac{w_1 w_2}{n}\sigma_1^2 G\theta_{21}^{\frac{1}{2}}.$$

3. Comparison of proportional-Neyman Allocation with proportional Allocation.

The function $D(\lambda, \theta_{21})$ produced by difference of variance of the estimate $\bar{y}_w$ of population mean under proportional allocation and proportional -Neyman allocation is defined by

$$D(\lambda, \theta_{21}) = [V(\bar{y}_w)_P - V(\bar{y}_w)_{P.N}] / \frac{w_1 w_2}{n}\sigma_1^2$$
$$= (1 + \theta_{21}))[I_0(P_{21}) + I_0(q_{21}^{(1)})] - G\theta_{21}^{\frac{1}{2}}[I_{-\frac{1}{2}}(P_{21}) + I_{-\frac{1}{2}}(q_{21}^{(1)})] \qquad (3.1)$$

where $I_i(x)=I_x(\frac{1}{2}f+i,\ \frac{1}{2}f+i)$, $q_{21}^{(1)}=1-P_{21}^{(1)}=\frac{1}{\lambda\theta_{21}+1}$

and $D(\lambda,\ \theta_{21})=D_P(\lambda,\ \theta_{21})-D_N(\frac{1}{\lambda'}\ \theta_{21})$

using [2], we obtain

$$D(\lambda,\ \theta_{21})=D_P(\lambda,\ \theta)-D_P(\frac{1}{\lambda}\theta_{21})+(1+\theta_{21}-G\theta_{21}^{\frac{1}{2}}) \quad (3.2)$$

Lemma 2. For any $\lambda\geq 1$, $\lim\limits_{\theta 21\to 1}D(\lambda,\ \theta_{21})\leq 0$.

Proof : For any $\lambda\geq 1$, $\lim\limits_{\theta 21\to 1}D_P(\lambda,\theta_{21})\leq 0$, $\lim\limits_{\theta 21\to 1}D_N(\lambda,\ \theta_{21})\geq 0$ by [2]

Form (3.2), we have

$$\lim_{\theta 21\to 1}D(\lambda,\ \theta_{21})=\lim_{\theta 21\to 1}D_P(\lambda,\ \theta_{21})-\lim_{\theta 21\to 1}D_N(\frac{1}{\lambda},\ \theta_{21})\leq 0 \qquad \text{Q. E. D.}$$

Next let us consider the case that is a fixed but arbitrary number.

Lemma 3. For any given $\lambda\geq 1$, $\frac{\partial}{\partial\theta_{21}}\ D(\lambda,\ \theta_{21})<0$, for $0<\theta_{21}<1$.

Proof : Form (3.1), we have

$$\frac{\partial}{\partial\theta_{21}}\ D(\lambda,\ \theta_{21})=[I_0(p_{21})+I_0(q_{21}^{(1)})]-\frac{1}{2}G\theta_{21}^{-\frac{1}{2}}[I_{-\frac{1}{2}}(P_{21})+I_{-\frac{1}{2}}(q_{21}^{(1)})]$$
$$+(1+\theta_{21})[R_0(P_{21})+R_0(q_{21}^{(1)})]-G\theta_{21}^{\frac{1}{2}}[R_{-\frac{1}{2}}(P_{21})+R_{-\frac{1}{2}}(q_{21}^{(1)})] \quad (3.3)$$

$$R_i(q_{21}^{(1)})=\frac{\partial}{\partial\theta_{21}}\ I_i(q_{21}^{(1)})=-P_{21}^{(1)\frac{1}{2}f+i}\,q_{21}^{(1)\frac{1}{2}f+i}/\theta_{21}B(\frac{1}{2}f+i,\ \frac{1}{2}f+i) \quad (3.4)$$

Substituting (3.4) in (3.3), we obtain

$$\frac{\partial}{\partial\theta_{21}}D(\lambda,\ \theta_{21})=[I_0(P_{21})-\frac{1}{2}G\theta_{21}^{-\frac{1}{2}}I_{-\frac{1}{2}}(P_{21})]+[I_0(\ q_{21}^{(1)})-\frac{1}{2}G\theta_{21}^{-\frac{1}{2}}\ I_{-\frac{1}{2}}(q_{21}^{(1)})]$$
$$+\lambda^{\frac{1}{2}f-\frac{1}{2}}\ \theta_{21}^{\frac{1}{2}f-1}\{(1+\theta_{21})\lambda^{\frac{1}{2}}[(\lambda+\theta_{21})^{-f}-(\lambda\theta_{21}+1)^{-f}]$$
$$-[(\lambda+\theta_{21})^{-f+1}-(\lambda\theta_{21}+1)^{-f+1}]\}/B(\frac{1}{2}f,\ \frac{1}{2}f) \quad (3.5)$$

It can be verified that for any x such that $0<x<1$,

$$I_0(x)-\frac{1}{2}G\theta_{21}^{-\frac{1}{2}}I_{-\frac{1}{2}}(x)\ <0 \quad (3.6)$$

Thus from (3.6), the first two terms in (3.5) are negative.

The third term in (3.5) can be written as

$$\frac{\lambda^{\frac{1}{2}f-\frac{1}{2}}\ \theta_{21}^{\frac{1}{2}f-1}\ (\lambda^{\frac{1}{2}}-1)[(\lambda\theta_{21}+1)^f(\theta_{21}-\lambda^{\frac{1}{2}})+(\lambda+\theta_{21})^f(\lambda^{\frac{1}{2}}\theta_{21}-1)]}{(\lambda+\theta_{21})^f(\lambda\theta_{21}+1)^f\ B(\frac{1}{2}f,\ \frac{1}{2}f)}$$

Since $(\lambda\theta_{21}+1)^f(\theta_{21}-\lambda^{\frac{1}{2}})+(\lambda+\theta_{21})^f(\lambda^{\frac{1}{2}}\theta_{21}-1)<(\lambda+1)^f(\lambda^{\frac{1}{2}}+1)(\theta_{21}-1)<0$, then the third term in (3.5), is olso negative. Hence, for $0 < \theta_{21} < 1$,

$$\frac{\partial}{\partial\theta_{21}} D(\lambda, \theta_{21}) < 0.$$

Lemma 4. For any given $\lambda \geq 1$ satisfying

$$1 - \frac{1}{2}G + (\lambda^{\frac{1}{2}f-\frac{1}{2}} + 1)(\lambda^{\frac{1}{2}} - 1)/\lambda^{\frac{1}{2}f} B(\frac{1}{2}f, \frac{1}{2}f) > 0 \qquad (3.7)$$

$\exists\, \theta' > 1$ such that $\frac{\partial}{\partial\theta_{21}} D(\lambda, \theta_{21}) > 0$. $\forall \theta_{21} > \theta'$.

Proof : From(3.5), $\frac{\partial}{\partial\theta_{21}} D(\lambda, \theta_{21})$ can also be written as

$$\frac{\partial}{\partial\theta_{21}} D(\lambda, \theta_{21}) = [I_0(P_{21}) + I_0(q_{21}^{(1)})] - \frac{1}{2}G\theta_{21}^{-\frac{1}{2}}[I_{-\frac{1}{2}}(P_{21}) + I_{-\frac{1}{2}}(q_{21}^{(1)})]$$
$$+ \lambda^{\frac{1}{2}f-\frac{1}{2}} \theta_{21}^{\frac{1}{2}f-1} \{(1+\theta_{21})\lambda^{\frac{1}{2}}[(\lambda+\theta_{21})^{-f} - (\lambda\theta_{21}+1)^{-f}]$$
$$-[(\lambda+\theta_{21})^{-f+1} - (\lambda\theta_{21}+1)^{-f+1}]\} / B(\frac{1}{2}f, \frac{1}{2}f).$$

It can be verified that, for $h \geq 0$, $\lim_{\theta_{21}\to\infty}[I_1(P_{21}) + I_1(q_{21}^{(1)})]/\theta_{21}^{h} = 0^+$.

Hence

$$\lim_{\theta_{21}\to\infty}\frac{1}{\theta_{21}}\frac{\partial}{\partial\theta_{21}} D(\lambda, \theta_{21}) = [1 - \frac{1}{2}G + (\lambda^{\frac{1}{2}f-\frac{1}{2}} + 1)(\lambda^{\frac{1}{2}} - 1)/\lambda^{\frac{1}{2}f} B(\frac{1}{2}f, \frac{1}{2}f)]\, 0^+ = 0^+$$

This implies that $\exists\, \theta'$ such that $\frac{\partial}{\partial\theta_{21}} D(\lambda, \theta_{21}) > 0$. $\forall\, \theta_{21} > \theta'$. Q. E. D.

Lemmas(3) and (4) tell us that, for a given λ, $D(\lambda, \theta_{21})$, as a function of θ_{21}, decreases first and then increases. Affer the above three lemmas (2, 3, and 4) are established, we are ready to prove the following theorem.

Theerem 1. For any given $\lambda \geq 1$, satistying (3.9), in (0.1) and $\theta_0^{(2)} > 1$ such that $D(\lambda, \theta_0^{(1)}) = D(\lambda, \theta_0^{(2)}) = 0$ and $D(\lambda, \theta_{21}) > 0$ $\forall\, \theta_{21} < \theta_0^{(1)}$ or $\theta_{21} > \theta_0^{(2)}$

Proof : Because of Lemmas (2, 3 and 4) it suffices to prove this theorem by showing that $D(\lambda, 0) > 0$ and $\lim_{\theta_{21}\to\infty} D(\lambda, \theta_{21}) > 0$

when $\theta_{21} = 0$. $P_{21} = 0$ and $q_{21}^{(1)} = 1$, thus $D(\lambda, 0) = 1 > 0$. It can be seen that $\lim_{\theta_{21}\to\infty} D(\lambda, \theta_{21})/\theta_{21} = 1$. This implies that $\lim_{\theta_{21}\to\infty} D(\lambda, \theta_{21}) > 0$. Q. E. D.

and the existence of $\theta_0^{(1)}$ and $\theta_0^{(2)}$ is in evidence.

For any given $\lambda \geq 1$, satisfying (3.7), the above theorem assures us that there exist $\theta_0^{(1)}$ in (0.1) and $\theta_0^{(2)} > 1$ such that for each $\theta_{21} < \theta_0^{(1)}$ or $\theta_{21} > \theta_0^{(2)}$ propotional-Neyman allocation is always more efficient than proportional allocation.

Taking, in particular, $\lambda = 1$, we obtain the following corollary to Theosem 1.

Corollary $D(1, \theta_{21}) > 0$ if $\theta_{21} < \theta_m^{(1)}$ or $\theta_{21} > \theta_m^{(2)}$ and $D(1, \theta_{21}) \leq O$ if $\theta_m^{(1)} \leq \theta_{21} \leq \theta_m^{(2)}$ where $\theta_m^{(1)} = \frac{1}{2}(G^2 - 2 - G\sqrt{G^2-4})$, $\theta_m^{(2)} = \frac{1}{2}(G^2 - 2 + G\sqrt{G^2-4})$ (3.8)

Proof : When $\lambda = 1$, $P_{21} = \theta_{21}/(1+\theta_{21})$ and $q_{21}^{(1)} = 1/(1+\theta_{21}) = q_{21}$.

Hence $I_{\!\!\iota}(P_{21}) + I_{\!\!\iota}(q_{21}^{(1)}) = 1$ and $D(1, \theta_{21}) = \theta_{21} - G\theta_{21}^{\frac{1}{2}} + 1$.

Therefore $D(1, \theta_{21}) = 0$ has two roots $\theta_m^{(1)}$ and $\theta_m^{(2)}$ defined in (3.8). Thus $D(1, \theta_{21}) > 0$ for $\theta_{21} < \theta_m^{(1)}$ or $\theta_{21} > \theta_m^{(2)}$ and $D(1, \theta_{21}) \leqq 0$ for $\theta_m^{(1)} \leqq \theta \leqq \theta_m^{(2)}$

Q. E. D.

If the set S is defined as $S = \{\theta_{21} : \theta_m^{(1)} \leqq \theta_{21} \leqq \theta_m^{(2)}\}$, then $D(1, \theta_{21}) \leqq 0$ if θ_{21} is in S and $D(1, \theta_{21}) > 0$ if θ_{21} is in S.

we shall now consider the case when θ_{21} is a fixed but arbitrary number.

Lemma 5. For any given $\theta_{21} \geqq 0$, $\exists \lambda'$ such that $\frac{\partial}{\partial\lambda} D(\lambda, \theta_{21}) > 0$, $\forall \lambda > \lambda'$

Proof : From (3.1), we have

$$\frac{\partial}{\partial\lambda} D(\lambda, \theta_{21}) = (1+\theta_{21})[Q_0(P_{21}) + Q_0(q_{21}^{(1)})] - G\theta_{21}^{\frac{1}{2}}[Q_{-\frac{1}{2}}(P_{21}) + Q_{-\frac{1}{2}}(q_{21}^{(1)})] \quad (3.9)$$

where $Q_i(P_{21})$ and $Q_i(q_{21}^{(1)})$ are as follows :

$$Q_i(P_{21}) = \frac{\partial}{\partial\lambda} I_{\!\!\iota}(P_{21}) = \frac{\partial}{\partial\lambda}\int_0^p x^{\frac{1}{2}f+i-1}(1-x)^{\frac{1}{2}f+i-1}\,dx \,/ B(\frac{1}{2}f+i, \frac{1}{2}-i)$$

$$= -P_{21}^{\frac{1}{2}f+i}\, q_{21}^{\frac{1}{2}f+i} / \lambda B(\frac{1}{2}f+i, \frac{1}{2}f+i)$$

$$Q_i(q_{21}^{(1)}) = \frac{\partial}{\partial\lambda} I_{\!\!\iota}(q_{21}^{(1)}) = -P_{21}^{(1)\frac{1}{2}f+i}\, q_{21}^{(1)\frac{1}{2}f+i} / \lambda B(\frac{1}{2}f+i, \frac{1}{2}f+i) \quad (3.10)$$

From (3.9) and (3.10), we obtain

$$\frac{\partial}{\partial\lambda} D(\lambda, \theta_{21}) = \theta_{21}^{\frac{1}{2}f}\, \lambda^{\frac{1}{2}f-\frac{3}{2}}(\lambda^{\frac{1}{2}} - 1)\left[\frac{\lambda^{\frac{1}{2}} - \theta_{21}}{(\lambda+\theta_{21})^f} + \frac{\lambda^{\frac{1}{2}}\theta_{21} - 1}{(\lambda\theta_{21}+1)^f}\right] / B(\frac{1}{2}f, \frac{1}{2}f)$$

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\frac{\partial}{\partial\lambda} D(\lambda, \theta_{21}) = [\theta_{21}^{\frac{1}{2}f}(1+\theta_{21}^{-f+1}) / B(\frac{1}{2}f, \frac{1}{2}f)]\ 0^+ = 0^f$$

Hence, $\exists\lambda'$ such that $\frac{\partial}{\partial\lambda} D(\lambda, \theta_{21}) > 0$. $\forall\lambda > \lambda'$

Q. E. D.

If we consider the limit of $\frac{\partial}{\partial\lambda} D(\lambda, \theta_{21})$, as λ tends to infinity, we get $\lim_{\lambda\to\infty}\frac{\partial}{\partial\lambda} D(\lambda, \theta_{21}) = 0^+$. *i. e.* $D(\lambda, \theta_{21})$ has a horizontal asymptote and $D(\lambda_0, \theta_{21})$ tends to zero. Thus for any given θ_{21} in S, there exists λ_0 such that $D(\lambda_0, \theta_{21}) = 0$ and $D(\lambda_0,$

$\theta_{21}) \geq 0$. For each $\lambda \leq \lambda_0$ the following theorem is then proved.

Theorm 2. For any given θ_{21} in S', $\exists \lambda_0$ such that $D(\lambda_0, \theta_{21}) = 0$ and $D(\lambda, \theta_{21}) \geq 0$. $\forall \lambda \leq \lambda_0$. That is, for any given θ_{21} in S', there exists λ_0 such that $D(\lambda_0, \theta_{21}) = 0$

proportional-Neyman allocation will always be more efficient than proportional allocation for each $\lambda < \lambda_0$.

4. Relative efficiency

Consider the relative efficiency of proportional-Neyman allocation with respect to proportional allocation

$$Q(\lambda, \theta_{21}) = V(\bar{y}_w)P / V(\bar{y}_w)_{P.N} = 1 / [1 - \frac{w_1 w_2}{w_1 + w_2\theta_{21}} D(\lambda, \theta_{21})] \qquad (4.1)$$

where, if $Q(\lambda, \theta_{21}) > 1$, proportional-Neyman allocation is more efficient than proportional allocation, and if $Q(\lambda, \theta_{21}) < 1$, proportional-Neyman allocation is less efficient than proportional allocation.

From (4.1), we also can see that $Q(\lambda, \theta_{21})$ will behave in the same manner as $D(\lambda, \theta_{21})$. Furthermore, some results of $Q(\lambda, \theta_{21})$ parallel to those of $D(\lambda, \theta_{21})$ which have been presented section 3 will be given below.

First, let us consider the case that λ is a fixed but arbitrary number. We have the following theorem for $Q(\lambda, \theta_{21})$.

Theorem 3. For any given $\lambda \geq 1$ satisfying (3.7), $\exists \theta_0^{(1)}$ in (0.1) and $\theta_0^{(2)} > 1$ such that $Q(\lambda, \theta_0^{(1)}) = Q(\lambda, \theta_0^{(2)}) = 1$ and $Q(\lambda, \theta_{21}) > 1$, $\forall \theta_{21} < \theta_0^{(1)}$ or $\theta_{21} > \theta_0^{(2)}$

Next we shall consider the case in which θ_{21} is a fixed but arbitrary number.

$Q(\lambda, \theta_{21})$ and $D(\lambda, \theta_{21})$ will behave exactly the same.

Then the behavior of $Q(\lambda, \theta_{21})$ as a function of λ, will be given in the following theorem.

Thorem 4. For any given Q_{21} in S', $\exists \lambda_0$ such that $Q(\lambda_0, \theta_{21}) = 1$ and $Q(\lambda, Q_{21}) \geq 1$ $\forall \lambda \leq \lambda$

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