FOLIATIONS WITH SINGULARITIES IN VIRTUAL BUNDLES

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0. Introduction

The theory of foliations has been studied for almost twenty years since about 1960, because of that the study of foliations is directly connected with characteristic class of manifolds.

As a result of those we had many beautiful properties with respect to differentiable manifolds([4], [5], [8], [9]).

The purpose of this paper is to prove our main theorem 3.5 and to describe many concepts of manifolds with almost complex structures, because of that foliations and almost complex structures have closed relations([7], [8]).

Let M be a complex analytic manifold with $\dim_{\mathbb{C}} M = m(\geqq 2)$ and $\mathcal{F}$ be a foliation of M with singular set S. A holomorphic subbundle F of $T = TM$ is integrable if $T\mathcal{F} \cong F$ (Definition 2.7). Theorem 3.5 says that the total Chern class of a virtual bundle $\zeta = \sum_{i=0}^{q} (-1)^i E_i$ by an invariant polynomial (Definition 2.2) does not depend on the choice of connections for E_i, where E_i $(i=0,1,\cdots,q)$ are smooth complex vector bundles over $U \subset M$. In order to prove this theorem we use a compact connected component Z of S which is a deformation retract of U and some lemmas (§ III).

In § I, we shall describe some properties of almost complex manifolds(Proposition 1.5 and 1.7) and the concepts of holomorphic bases and anti-holomorphic bases in details (Example 1.6).

In § II, the concepts of connection, foliation and sheaf which need in § III are described (Definition 2.1, 2.7 and 2.10) and some properties with respect to those (Proposition 2.4, 2.5, 2.6, 2.8 and 2.9) will be proved.

1. Almost Complex Structures

Let V be a real vector space. A complex structure on V is an endomorphism

$J: V \to V$ with $J^2 = -1$. If V has a complex structure J, by defining

$$(a+ib)x = ax + bJ(x)$$

for $x \in V$ and $(a+ib) \in C$ (complex numbers), V becomes a complex vector space. Conversely, if V is a complex vector space, it has a complex structure

$$J: V \to V\ (x \mapsto J(x) = ix).$$

The canonical complex structure of R^{2n} (R: real numbers) is given by the matrix

$$J_0 = \begin{pmatrix} O, & I_n \\ -I_n, & O \end{pmatrix}$$

where I_n denotes the identity matrix of degree n. We shall denote a real vector space V with a complex structure by $V(J)$.

PROPOSITION 1.1. Let f be a real linear map from $V(J)$ to $V'(J')$. In a natural manner, if we consider V and V' as complex vector spaces, then f is a complex linear map if and only if $J' \circ f = f \circ J$.

PROOF. For $a+ib \in C$, $x \in V$ and $x' \in V'$ since

$$(a+ib)x = (a+bJ)x \text{ and } (a+ib)x' = (a+bJ')x',$$

$$f((a+ib)x) = (a+ib)f(x) \Longleftrightarrow af(x) + bf(J(x)) = af(x) + bJ'f(x)$$

$$\Longleftrightarrow f \circ J = J' \circ f. \qquad \text{Q. E. D.}$$

For $R^{2n}(J_0)$ and $(x^1, \cdots, x^n, y^1, \cdots y^n) \in R^{2n}$, we have to note that

(i) $J_0((x^1, \cdots, x^n, y^1, \cdots, y^n))$

$$= \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} \begin{pmatrix} x^1 \\ \vdots \\ x^n \\ y^1 \\ \vdots \\ y^n \end{pmatrix} = \begin{pmatrix} y^1 \\ \vdots \\ y^n \\ -x^1 \\ \vdots \\ -x^n \end{pmatrix} = (y^1, \cdots, y^n, -x^1, \cdots, -x^n),$$

(ii) There exist $x_1, \cdots, x_n \in R^{2n}$ such that $\{x_1, \cdots, x_n, J_0x_1, \cdots, J_0x_n\}$ is a base of R^{2n}.

(iii) $GL(n, C) \cong$ the subgroup of $GL(2n, R)$ consisting of matrices which commute with the matrix $J_0 = \widetilde{GL}(2n, R)$.

PROOF. Note that

$$\begin{array}{ccc} C^n & \longrightarrow & R^{2n} \\ \Downarrow & & \Downarrow \\ (a_1+ib_1,\ \cdots,\ a_n+ib_n) & \leadsto & (a_1,\ \cdots,\ a_n,\ -b_1,\ \cdots,\ -b_n) \end{array} \qquad \text{(A)}$$

and for any $\begin{pmatrix} a_{11}+ib_{11}, & \cdots\cdots, & a_{1n}+ib_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1}+ib_{n1}, & \cdots\cdots, & a_{nn}+ib_{nn} \end{pmatrix} \in GL(n, C)$ and

$(\alpha_1+i\beta_1,\ \cdots,\ \alpha_n+i\beta_n)\in C^n$,

$$\begin{pmatrix} a_{11}+ib_{11}, & \cdots, & a_{1n}+ib_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1}+ib_{n1}, & \cdots, & a_{nn}+ib_{nn} \end{pmatrix}\begin{pmatrix} \alpha_1+i\beta_1 \\ \cdots \\ \alpha_n+i\beta_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n}(a_{1j}\alpha_j-b_{1j}\beta_j)+i\sum_{j=1}^{n}(a_{1j}\beta_j+b_{1j}\alpha_j) \\ \cdots \\ \sum_{j=1}^{n}(a_{nj}\alpha_j-b_{nj}\beta_j)+i\sum_{j=1}^{n}(a_{nj}\beta_j+b_{nj}\alpha_j) \end{pmatrix}$$

Since by (A)

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{1j}\alpha_j-b_{1j}\beta_j)+i\sum_{j=1}^{n}(a_{1j}\beta_j+b_{1j}\alpha_j) \\ \cdots \\ \sum_{j=1}^{n}(a_{nj}\alpha_j-b_{nj}\beta_j)+i\sum_{j=1}^{n}(a_{nj}\beta_j+b_{nj}\alpha_j) \end{pmatrix} \leadsto \begin{array}{l} (\sum_{j=1}^{n}(a_{1j}\alpha_j-b_{1j}\beta_j),\ \cdots,\ \sum_{j=1}^{n}(a_{nj}\alpha_j-b_{nj}\beta_j), \\ -\sum_{j=1}^{n}(a_{1j}\beta_j+b_{1j}\alpha_j),\cdots,-\sum_{j=1}^{n}(a_{nj}\beta_j+b_{nj}\alpha_j))\in R^{2n}, \end{array}$$

if we put

$$A=\begin{pmatrix} a_{11}, \cdots, a_{1n} \\ \cdots\cdots \\ a_{n1}, \cdots, a_{nn} \end{pmatrix}, \qquad B=\begin{pmatrix} b_{11}, \cdots, b_{1n} \\ \cdots\cdots \\ b_{n1}, \cdots, b_{nn} \end{pmatrix}$$

then

$$\begin{array}{ccc} GL(n, C) & \longrightarrow & GL(2n, R) \\ \Downarrow & & \Downarrow \\ A+iB & \leadsto & \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \end{array}$$

because of that

$$\begin{pmatrix} A & B \\ & \\ -B & A \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ -\beta_1 \\ \vdots \\ -\beta_n \end{pmatrix} = (\sum_{j=1}^{n}(a_{1j}\alpha_j-b_{1j}\beta_j),\ \cdots,\sum_{j=1}^{n}(a_{nj}\alpha_j-b_{nj}\beta_j),\ -\sum_{j=1}^{n}(a_{1j}\beta_j+b_{1j}\alpha_j),\cdots,\ -\sum_{j=1}^{n}(a_{nj}\beta_j+b_{nj}\alpha_j)).$$

Next,

$$\begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} -B & A \\ -A & -B \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}.$$

Conversely, for any matrices A_1, A_2, B_1, B_2 with degree n

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}\begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix} = \begin{pmatrix} -B_1 & A_1 \\ -B_2 & A_2 \end{pmatrix} = \begin{pmatrix} A_2 & B_2 \\ -A_1 & -B_1 \end{pmatrix} = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$$

implies that $A_2=-B_1$ and $A_1=B_2$. Therefore, the matrices which are of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad (A,\ B: \text{matrices with degree } n)$$

only commute with J_0.

Finally, for $A_1+iB_1,\ A_2+iB_2 \in GL(n, C)$

$$(A_1+iB_1)\ (A_2+iB_2)=(A_1A_2-B_1B_2)+i(A_1B_2+B_1A_2)$$

and
$$\begin{pmatrix} A_1 & B_1 \\ -B_1 & A_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ -B_2 & A_2 \end{pmatrix} = \begin{pmatrix} A_1A_2-B_1B_2 & A_1B_2+B_1A_2 \\ -(A_1B_2+B_1A_2) & A_1A_2-B_1B_2 \end{pmatrix}$$

and thus

$$\begin{array}{cccc} \varphi: & GL(n, C) & \longrightarrow & \widetilde{GL}(2n, R) \\ & \cup\!\!\!| & & \cup\!\!\!| \\ & A+iB & \longmapsto & \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \end{array}$$

is an isomorphism. Q. E. D.

The coset represented by $S\in GL(2n, R)$ of $GL(2n, R)/\widetilde{GL}(2n, R)$ $(=GL(2n, R)/GL(n, C))$ corresponds to the complex structure SJ_0S^{-1} of R^{2n}. For a complex structure J of R^{2n} there exists a unique element $[S]\in GL(2n, R)/\widetilde{GL}(2n, R)$ such that $J=SJ_0S^{-1}$ by the following reasons: By (ii) above, for the complex structure J there is a basis $\{e_1', \cdots, e_n', Je_1', \cdots, Je_n'\}$ of R^{2n}. If $\{e_1, \cdots, e_n, J_0e_1, \cdots, J_0e_n\}$ is the base of R^{2n} for J_0 there exists an element $S\in GL(2n, R)$ such that

$$Se_k=e_k',\quad SJ_0e_k=Je_k'=JSe_k.$$

Thus $J=SJ_0S^{-1}$. If $S\in \widetilde{GL}(2n, R)$, $J=J_0$, i. e., $J\neq J_0 \Longrightarrow S\notin \widetilde{GL}(2n, R)$. If $[S_1]\neq[S_2]$ in $GL(2n, R)/\widetilde{GL}(2n, R)$, we have $S_1J_0S_1^{-1}\neq S_2J_0S_2^{-1}$, because of that

$$S_1J_0S_1^{-1}=S_2J_0S_2^{-1} \Longrightarrow (S_2^{-1}S_1)J_0(S_1^{-1}S_2)=J_0 \Longrightarrow S_2^{-1}S_1\in \widetilde{GL}(2n, R).$$

Since $[S_2^{-1}S_1]\in GL(2n, R)/\widetilde{GL}(2n, R)$ it follows that $S_1J_0S_1^{-1}\neq S_2J_0S_2^{-1}$.

In consequence, we have

{the set of all complex structures of $R^{2n}\}\xleftrightarrow{1:1} GL(2n, R)/\widetilde{GL}(2n, R)$ as sets.

Let $V(J)$ be a real vector space and V^* its dual space. Then, the vector space $V^*(J)$ is defined as follows:

$$\langle Jx, x^* \rangle = \langle x, Jx^* \rangle \text{ for } x \in V \text{ and } x^* \in V^*,$$

where $\langle Jx, x^* \rangle = x^*(Jx)$. Also, for a real vector space V we put

$$V^c = V \otimes_R C,$$

which is called the complexification of V.

Let us assume that V is a real $2n$-dimensional vector space with a complex structure J. Then J can be uniquely extended to a complex structure of V^c, and the extended endomorphism, denoted also by J, as follows :

$$\forall x \in V \text{ and } a+ib \in C,$$
$$J(x \otimes (a+ib)) = Jx \otimes (a+ib).$$

In this case, since $J^2 = -1$ the eigenvalues of J are i and $-i$, where

$$-1 = \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix} \in GL(2n, R).$$

DEFINITION 1.2. We define that

$$V^{1,0} = \{z \in V^c \mid Jz = iz\} \text{ and } V^{0,1} = \{z \in V^c \mid Jz = -iz\}.$$

We also define the complex conjugation in V^c to be a real linear map

$$\begin{array}{ccc} V^c & \longrightarrow & V^c \\ \Psi & & \Psi \\ x \otimes (a+ib) & \longmapsto & x \otimes (a-ib). \end{array}$$

PROPOSION 1.3. (i) $V^{1,0} = \{x - iJx \mid x \in V\}$ and $V^{0,1} = \{x + iJx \mid x \in V\}$

(ii) $V^c = V^{1,0} \oplus V^{0,1}$ (as complex vector spaces)

(iii) Let φ be the complex conjugation in V^c. Then

$$\varphi \mid V^{1,0} : V^{1,0} \cong V^{0,1}.$$

PROOF. At first, we have to note that $V^c = \{x + iy \mid x, y \in V\}$.

$$J(x - iJx) = Jx + ix = i(x - iJx)$$
$$J(x + iJx) = Jx - ix = -i(x + iJx),$$

and thus (i) is proved. For an element $x \in V^c$

$$Jx = ix \Longrightarrow Jix = -x = i(ix),$$

i. e., $x \in V^{1,0} \Longrightarrow ix \in V^{1,0}$. Similarly, $x \in V^{0,1} \Longrightarrow ix \in V^{0,1}$.

If $x (\neq 0) \in V^{1,0} \cap V^{0,1}$, $Jx = ix = -ix \Longrightarrow 2ix = 0 \Longrightarrow x = 0$, *i. e.*, $V^{1,0} \cap V^{0,1} = \{0\}$. For any element $x + iy \in V^c$, since

$$x+iy=\frac{1}{2}\{(x+iy)-(iJx-Jy)+(iJx-Jy)+(x+iy)\}$$
$$=\frac{1}{2}\{[(x-iJx)+i(y-iJy)]+[(x+iJx)+i(y+iJy)]\}\in V^{1,0}\oplus V^{0,1},$$

$V^c=V^{1,0}\oplus V^{0,1}$ (as complex vector spaces) is true.

In the diagram

$$\begin{array}{ccc} x-iJx & \longmapsto & x+iJx \\ \cap & & \cap \\ V^{1,0} & \xrightarrow{\ \varphi\,|\,V^{1,0}\ } & V^{0,1} \\ \cup & & \cup \\ x-iJx & \longleftarrow & x+iJx \end{array} \quad \text{for } x, Jx\in V,$$

$\varphi\ |V^{1,0}:V^{1,0}\longrightarrow V^{0,1}$ is an isomorphism. Q. E. D.

Similarly, if we put

$$V_{1,0}=\{x^*\in V^{*c}\mid \forall\, x\in V^{0,1}\ \langle x, x^*\rangle=0\}$$
$$V_{0,1}=\{x^*\in V^{*c}\mid \forall\, x\in V^{1,0}\ \langle x, x^*\rangle=0\},$$

it is easily proved that $V^{*c}=V_{1,0}\oplus V_{0,1}$ as complex vector spaces.

DEFINITION 1.4. Let M be a real smooth manifold and $T(M)$ the total space of the tangent bundle over M. A tensor field $J: T(M)\longrightarrow T(M)$ ([10]) is an almost complex structure on M is an endomorphism of the tangent space $T_x(M)$ for all $x\in M$ such that $J^2=-1$, where

$$1=\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\in GL(2n, R)$$

and $\dim_R M=2n\,(n=0, 1, 2, \cdots)$. A real smooth manifold M with a fixed almost complex structure is called an almost complex manifold.

PROPOSITION 1.5. Every almost complex manifold is orientable.

PROOF. For a point $x\in M$, $T_x(M)$ becomes a complex vector space with dimension n by the given complex structure J. Suppose two bases $\{z_1, \cdots, z_n\}$ and $\{z_1', \cdots, z_n'\}$ of $T_x(M)$ as a complex vector space. Then $\{z_1, \cdots, z_n, Jz_1, \cdots, Jz_n\}$ and $\{z_1', \cdots z_n', Jz_1', \cdots, Jz_n'\}$ are bases of $T_x(M)$ as a real $2n$-dimensional vector space. In this case,

$$z_1'=a_{11}z_1+\cdots+a_{1n}z_n+a_{1n+1}Jz_1+\cdots+a_{12n}Jz_n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$z_n' = a_{n1}z_1 + \cdots + a_{nn}z_n + a_{nn+1}Jz_1 + \cdots + a_{n2n}Jz_n$$
$$Jz_1' = -a_{1n+1}z_1 - \cdots - a_{12n}z_n + a_{11}Jz_1 + \cdots + a_{1n}Jz_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Jz_n' = -a_{nn+1}z_1 - \cdots - a_{n2n}z_n + a_{n1}Jz_1 + \cdots + a_{nn}Jz_n$$

where $a_{ij} \in \boldsymbol{R}$ for $1 \leqq i,\ j \leqq 2n$. If we put

$$A = \begin{pmatrix} a_{11} \cdots\cdots a_{1n} \\ \cdots\cdots\cdots\cdots \\ a_{n1} \cdots\cdots a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} a_{1n+1} \cdots\cdots a_{12n} \\ \cdots\cdots\cdots\cdots\cdots \\ a_{nn+1} \cdots\cdots a_{n2n} \end{pmatrix},$$

then

$$\begin{pmatrix} z_1' \\ \vdots \\ z_n' \\ Jz_1' \\ \vdots \\ Jz_n' \end{pmatrix} = \begin{pmatrix} A & B \\ & \\ & \\ -B & A \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \\ Jz_1 \\ \vdots \\ Jz_n \end{pmatrix}$$

and thus

$$\begin{Vmatrix} A & B \\ -B & A \end{Vmatrix} = \|A\|^2 + \|B\|^2 > 0,$$

where $\|A\|$ is the determinant of A. This means that $T_x(M)$ has only one orientation determined by $\{z_1, \cdots, z_n, Jz_1, \cdots, Jz_n\}$. Therefore, for each $x \in M$, we define the orientation of $T_x(M)$ by $\{x_1, \cdots, x_n, Jx_1, \cdots, Jx_n\}$. For an open neighborhood U of x such that $h: U \times \boldsymbol{R}^{2n} \xrightarrow{\cong} T_u(M)$,

where $T_u(M)$ is the total space of $\tau_M | U$ and τ_M the tangent bundle of M,

$$\{h(x, e_1), \ \cdots, \ h(x, e_{2n})\} = \{z_1, \ \cdots, \ z_n\} \quad (x \in U)$$

are bases of $T_x(M)$ as a real and complex vector space, respectively. Thus, h preserves orientation for each $x \in U$. Q. E. D.

Note that the orientation of an almost complex manifold M given in the proof above is called the natural orientation.

EXAMPLE 1.6. Let M be a complex analytic manifold with dimension n. For each chart $(U_\alpha, \varphi_\alpha)$ of M

$$\begin{array}{ccccc} U_\alpha & \xrightarrow{\ \varphi_\alpha\ } & \boldsymbol{C}^n & \xrightarrow{\ \psi\ } & \boldsymbol{R}^{2n} \\ \Cup & & \Cup & & \Cup \\ x & \longmapsto & (z^1, \cdots, z^n) & \longmapsto & (x^1, \cdots, x^n, y^1, \cdots, y^n), \end{array}$$

where $z^k=x^k+iy^k(k=1,2,\cdots,n)$, makes M into a real $2n$-dimensional smooth manifold, written M_R. This is clear by noting that for $U_\alpha\cap U_\beta\neq\phi$
$\varphi_\beta\varphi_\alpha^{-1}|\varphi_\alpha(U_\alpha\cap U_\beta):\varphi_\alpha(U_\alpha\cap U_\beta)\longrightarrow\varphi_\beta(U_\alpha\cap U_\beta)$ is holomorphic and

$$\frac{\partial}{\partial z^k}=\frac{1}{2}\left(\frac{\partial}{\partial x^k}-i\frac{\partial}{\partial y^k}\right)\quad(k=1,2,\cdots,n).$$

Let τ_M be the tangent bundle over M, i.e., $\tau_M=(T(M),\ \pi,\ M)$. Then, for each $x\in M$ $T_x(M)$ has a holomorphic base $\{\frac{\partial}{\partial z^1},\cdots,\frac{\partial}{\partial z^n}\}$, where if $x\in U_\alpha$ $\varphi_\alpha(x)=(z^1,\cdots,z^n)\in C^n$. We can also consider the real tangent bundle $\tau_{M_R}=(T(M_R),\ \pi_R,\ M_R)$. For each $x\in M$, $T_x(M_R)$ is a real $2n$-dimensional vector space with a base $\{\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^n},\frac{\partial}{\partial y^1},\cdots,\frac{\partial}{\partial y^n}\}$, where $z^k=x^k+iy^k\ (k=1,2,\cdots,n)$. Therefore, we can introduce an almost complex structure J into $T(M_R)$ by

$$J\left(\frac{\partial}{\partial x^j}\right)=\frac{\partial}{\partial y^j},\qquad J\left(\frac{\partial}{\partial y^j}\right)=-\frac{\partial}{\partial x^j}$$

for $j=1,2,\cdots,n$. As the above statements $T_x(M_R)^c=T_x(M_R)\otimes_R C$ has the complex structure J such that $T_x(M_R)(J)\otimes_R C$. Define

$$T_x(M_R)^{1,0}=\{z\in T_x(M_R)^c\mid Jz=iz\}$$
$$T_x(M_R)^{0,1}=\{z\in T_x(M_R)^c\mid Jz=-iz\},$$

then by Proposition 1.3, we have

(i) $T_x(M_R)^{1,0}=\{y-iJy\mid y\in T_x(M_R)\}$, $T_x(M_R)^{0,1}=\{y+iJy\mid y\in T_x(M_R)\}$

(ii) $T_x(M_R)^c=T_x(M_R)^{1,0}\oplus T_x(M_R)^{0,1}$ (as complex vector spaces)

(iii) $T_x(M_R)^{1,0}\cong T_x(M_R)^{0,1}$ by the complex conjugation.

PROPOSITION 1.7. Under the above situation,

(i) $\{\frac{\partial}{\partial z^1},\cdots,\frac{\partial}{\partial z^n}\}$ is a base of $T_x(M_R)^{1,0}$,

(ii) $\{\frac{\partial}{\partial \bar z^1},\cdots,\frac{\partial}{\partial \bar z^n}\}$ is a base of $T_x(M_R)^{0,1}$.

PROOF. For $z^k=x^k+iy^k$, since

$$\frac{\partial}{\partial z^k}=\frac{1}{2}\left(\frac{\partial}{\partial x^k}-i\frac{\partial}{\partial y^k}\right),\ \frac{\partial}{\partial \bar z^k}=\frac{1}{2}\left(\frac{\partial}{\partial x^k}+i\frac{\partial}{\partial y^k}\right)$$

$J\left(\frac{\partial}{\partial z^k}\right)=i\frac{\partial}{\partial z^k}$ and $J\left(\frac{\partial}{\partial \bar z^k}\right)=-i\frac{\partial}{\partial \bar z^k}$. It follows that

$$J\begin{pmatrix}\frac{\partial}{\partial z^1}\\ \vdots \\ \frac{\partial}{\partial z^n}\end{pmatrix}=\frac{1}{2}J\begin{pmatrix}\frac{\partial}{\partial x^1}\\ \vdots\\ \frac{\partial}{\partial x^n}\\ -\frac{\partial}{\partial y^1}\\ \vdots\\ -\frac{\partial}{\partial y^n}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\frac{\partial}{\partial y^1}\\ \vdots\\ \frac{\partial}{\partial y^n}\\ \frac{\partial}{\partial x^1}\\ \vdots\\ \frac{\partial}{\partial x^n}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\frac{\partial}{\partial y^1}+i\frac{\partial}{\partial x^1}\\ \vdots\\ \frac{\partial}{\partial y^n}+i\frac{\partial}{\partial x^n}\end{pmatrix}=i\begin{pmatrix}\frac{\partial}{\partial z^1}\\ \vdots\\ \frac{\partial}{\partial z^n}\end{pmatrix}$$

and

$$J\begin{pmatrix}\frac{\partial}{\partial \bar z^1}\\ \vdots\\ \frac{\partial}{\partial \bar z^n}\end{pmatrix}=-i\begin{pmatrix}\frac{\partial}{\partial \bar z^1}\\ \vdots\\ \frac{\partial}{\partial \bar z^n}\end{pmatrix}.$$

Q. E. D.

We put

$$\tau=T(M_R)\otimes_R C=T\oplus \bar T,\quad \tau^*=T^*\oplus \bar T^*,$$

where

$$T=\bigcup_{x\in M}T_x(M_R)^{1,0}\ \text{and}\ \bar T=\bigcup_{x\in M}T_x(M_R)^{0,1}.$$

In this case, T is called the holomorphic tangent bundle of M, and $\bar T$ is called the anti-holomorphic tangent bundle of M.

Let U be an open subset of M. On U, let $z^1,\cdots,z^n$ be a complex analytic coordinate system, then by Proposition 1.7,

$\{\frac{\partial}{\partial z^1},\cdots,\frac{\partial}{\partial z^n}\}$ is a holomorphic frame of T,

$\{dz^1,\cdots,dz^n\}$ is a holomorphic frame of T^*.

2. Connections, Foliations and Sheaves

Let M be a smooth complex manifold such that $2m=\dim_R M_R$ $(m\geq 2)$ and let E *be* a smooth complex vector bundle over M. That is, for each $x\in M$ there exists an open neighborhood U of x such that there exists a homeomorphism

$$h: U\times C^n \longrightarrow \pi^{-1}(U)$$

such that $h\,|\,x\times C^n: x\times C^n \longrightarrow \pi^{-1}(x)$ is of the C^∞-class

$(h(x,(z^1,\cdots,z^n))=(z'^1,\cdots,z'^n)\Longrightarrow \exists\, j,k \cdot\ni\cdot \frac{\partial z'^j}{\partial \bar z^k}\neq 0)$, where $\dim_C \pi^{-1}(x)=n$.

Under the above situation we put

(i) $\tau = T(M_R) \otimes_R C = T(M_R)^c$

(ii) $\Gamma_\infty(E)$ = the space of all smooth sections of $\hat{E}$

(iii) E^* = the bundle over M which is dual of $E = \mathrm{Hom}_R(E, C)$

(iv) $\wedge^i E$ = the i-th exterior power of E.

DEFINITION 2.1. Let E be a smooth complex vector bundle over M which is a smooth real manifold with $\dim_R M_R = 2m$. A connection ∇ for E is a complex linear map from $\Gamma_\infty(E)$ to $\Gamma_\infty(\tau^* \otimes E)$ such that

$$\nabla(fs) = df \otimes s + f\nabla s,$$

where $f \in \{f : M \longrightarrow C \mid f \text{ is smooth}\} = A^0$ and $s \in \Gamma_\infty(E)$ (Note that f is not always satisfying the Cauchy-Riemann equations).

In consequence, since the connection ∇ is a local operator, it makes sense to talk about the restriction of ∇ to an open subset of M. Let U be an open subset of M. If $s \in \Gamma_\infty(E)$ vanishes on U, then ∇s is also zero on U. It follows from this remark that ∇ restricts to give a connection for $E|U$,

i. e.,

$$\nabla | U : \Gamma_\infty(E|U) \longrightarrow \Gamma_\infty(\tau^* \otimes E|U)$$

is defined. Accordingly, on U let $e_1, \cdots\cdots, e_n$ be a smooth frame of E the

$$\nabla e_i = \sum_{j=1}^{n} \theta_{ij} \otimes e_j,$$

where θ_{ij} is an 1-form, *i. e.*, $\theta_{ij} \in \Gamma_\infty(\wedge^1 \tau^*)$. In this case, the matrix $\theta = (\theta_{ij})$ is called the connection matrix of ∇ with respect to the frame $e_1, \cdots\cdots, e_n$.

$$\begin{aligned}
\nabla(\nabla e_i) &= \nabla\Big(\sum_{j=1}^{n} \theta_{ij} \otimes e_j\Big) = \sum_{j=1}^{n} \nabla(\theta_{ij} \otimes e_j) \\
&= \sum_{j=1}^{n} (d\theta_{ij} \otimes e_j - \theta_{ij} \nabla e_j) \\
&= \sum_{j=1}^{n} \Big(d\theta_{ij} \otimes e_j - \theta_{ij}\Big(\sum_{k=1}^{n} \theta_{jk} \otimes e_k\Big)\Big) \\
&= K_{i1} \otimes e_1 + \cdots\cdots + K_{in} \otimes e_n = \sum_{j=1}^{n} K_{ij} \otimes e_j,
\end{aligned}$$

where $K_{ij} = d\theta_{ij} - \sum_{k=1}^{n} \theta_{ik} \wedge \theta_{kj} \in \Gamma_\infty(\wedge^2 \tau^*)$. We put $K = (K_{ij})$, which is called the curvature matrix of ∇ with respect to $e_1, \cdots\cdots, e_n$. If $e_1', \cdots\cdots, e_n'$ is another smooth frame of E on U, let $A = (a_{ij})$ be determined by

$$e_i' = \sum_{j=1}^{n} a_{ij} e_j,$$

then $A \in GL(n, C)$. Then, we have

$$K' = AKA^{-1},$$

where K' is the curvature matrix of ∇ with respect to $e_1', \cdots\cdots, e_n'$.

DEFINITION 2.2. Let $M_n(C)$ be the algebra consisting of all $n \times n$ complex matrices. An <u>invariant polynomial</u> on $M_n(C)$ is a function

$$\varphi : M_n(C) \longrightarrow C$$

which is a complex polynomial in the entries of the matrix, and satisfies

$$\varphi(XY) = \varphi(YX) \quad (X, Y \in M_n(C)),$$

or equivalently

$$\varphi(TXT^{-1}) = \varphi(X)$$

for every non-singular matrix $T \in M_n(C)$. The trace function (a_{ij}) $(\in M_n(C)) \mapsto \sum_{i=1}^{n} a_{ii}$ and the determinant function $(a_{ij}) \mapsto \| a_{ij} \|$ are well-known examples of invariant polynomials on $M_n(C)$. Therefore, we have

$$\varphi(K') = \varphi(K).$$

Let $\sigma_j(K)$ denote the j-th elementary symmetric function of the eigenvalues of K, so that

$$\| 1 + tK \| = \begin{Vmatrix} 1 + tK_{11} \cdots\cdots\cdots\cdots tK_{1n} \\ \cdots\cdots\cdots\cdots \\ tK_{n1} \cdots\cdots\cdots\cdots 1 + tK_{nn} \end{Vmatrix}$$
$$= 1 + t\sigma_1(K) + \cdots\cdots\cdots + t^n \sigma_n(K).$$

(Note that

$$\| 1x - K \| = x^n - \sigma_1(K) x^{n-1} + \sigma_2(K) x^{n-2} - \cdots\cdots + (-1)^n \sigma_n(K).)$$

Thus $\sigma_j(K)$ is a $2j$-form, *i.e.*, $\sigma_j(K) \in \Gamma_\infty(\wedge^{2j} \tau^*)$. It is clear that if $n < m$, then $\sigma_j(K) = 0$ whenever $n < j \leqq m$. Since the eigenvalues of K coincide with the one of $AKA^{-1} = K'$, we have $\sigma_j(K) = \sigma_j(K')$ for $j = 1, 2, \cdots\cdots, n$. Furthermore, it is well-known that $d\sigma_j(K(\nabla)) = 0$, where $K(\nabla) : E \longrightarrow \wedge^2 \tau^* \otimes E$ defined by $K(\nabla) e_i = \sum_{j=1}^{n} K_{ij} \otimes e_j$ ([6], [11]).

Let $c_1(E), \cdots\cdots, c_m(E)$ be the Chern classes of E taken in the de Rham cohomology ring $H^*(M; C)$, then if $n < m$ and $n < j \leqq m$, $c_j(E) = 0$.

By the Chern-Weil theory of characteristic class in [6],

$[\sigma_j(K(\nabla))]$ (=the cohomology class of $\sigma_j(K(\nabla))$ in $H^*(M;C)) = (2\pi/\sqrt{-1})^j c_j(E)$

for $j = 1, 2, \cdots\cdots, n$. In particular, if $\widetilde{\nabla}$ is another connection for E, then

$$[\sigma_j(K(\nabla))] = [\sigma_j(K(\widetilde{\nabla}))].$$

Since any invariant polynomial on $M_n(C)$ can be expressed as a polynomial function of $\sigma_1, \cdots\cdots, \sigma_n$ ([11]), if φ is an invariant homogeneous polynomial of degree l we may set

$$\varphi = \widetilde{\varphi}(\sigma_1, \cdots\cdots \sigma_l)$$

which is a polynomial of $\sigma_1, \cdots\cdots, \sigma_l$. Define $\varphi(E) \in H^{2l}(M : C)$ by

$$\varphi(E) = \widetilde{\varphi}(c_1(E), \cdots\cdots, c_l(E)),$$

then we have the following ([3]):

(i) $\varphi(K(\nabla)) = \widetilde{\varphi}(\sigma_1(K), \cdots\cdots, \sigma_l(K))$

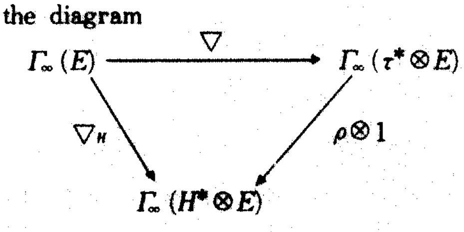
(ii) $d\varphi(K(\nabla)) = 0$

(iii) $[\varphi(K(\nabla))] = (2\pi/\sqrt{-1})^l \varphi(E)$.

DEFINITION 2.3. Let H be a smooth sub-vector bundle of τ, *i.e.*, there exists a a smooth sub-vector bundle $H^{\perp}$ of τ such that $\tau = H \oplus H^{\perp}$. Then $H^* \cong (\tau/H^{\perp})^*$ which is a quotient bundle of τ^*. Therefore there is a unique projection $\rho : \tau^* \longrightarrow H^*$ which is onto. A partial connection for E is a pair (H, ∇_H) where ∇_H is a complex linear from $\Gamma_\infty(E)$ into $\Gamma_\infty(H^* \otimes E)$ such that

$$\nabla_H(fs) = \rho(df) \otimes s + f\nabla_H(s)$$

for all $f \in A^0$ and $s \in \Gamma_\infty(E)$. For a partial connection (H, ∇_H) and a connection ∇ for E, if the diagram



is commutative then we say that ∇ is an extension of ∇_H.

PROPOSITION 2.4. Every partial connection (H, ∇_H) for E has its extension.

PROOF. Let $\{U_\alpha, \psi_\alpha\}_{\alpha \in \Lambda}$ be the atlas of M. Then, on each U_α there is a smooth frame $e_1^\alpha, \cdots\cdots, e_n^\alpha$ of E, and we have $\gamma_{ij}^\alpha \in \Gamma_\infty(H^*|U_\alpha)$ such that

$$\nabla_H e_i^\alpha = \sum_{j=1}^{n} \gamma_{ij}^\alpha \otimes e_j^\alpha.$$

We take an element θ_{ij}^α in $\rho^{-1}(\gamma_{ij}^\alpha) \in \Gamma_\infty(\tau^* | U_\alpha)$ and define a connection $\nabla | U_\alpha$ for $E | U_\alpha$ such that

$$(\nabla | U_\alpha) e_i^\alpha = \sum_{j=1}^{n} \theta_{ij}^\alpha \otimes e_j^\alpha.$$

On U_α it is clear that the diagram

$$\begin{array}{ccc} \Gamma_\infty(E | U_\alpha) & \xrightarrow{\ \nabla | U_\alpha\ } & \Gamma_\infty(\tau^* \otimes E | U_\alpha) \\ & {\scriptstyle \nabla_H} \searrow \quad \swarrow {\scriptstyle \rho \otimes 1} & \\ & \Gamma_\infty(H^* \otimes E | U_\alpha) & \end{array}$$

is commutative. Since M is paracompact, there is a partition of unity subordinate $\{\lambda_\alpha\}_{\alpha \in A}$ to the cover $\{U_\alpha\}_{\alpha \in A}$ of M, and thus the connection

$$\nabla = \sum_{\alpha \in A} \lambda_\alpha (\nabla | U_\alpha)$$

is an extension of ∇_H. Q. E. D.

For each $u \in \Gamma_\infty(H)$ we define

$$u : \Gamma_\infty(H^*) \longrightarrow A^0$$

by $u(s) = s(u)$, where $s \in \Gamma_\infty(H^*)$. Then we can also define

$$u : \Gamma_\infty(H^* \otimes E) \longrightarrow \Gamma_\infty(E)$$

by $u(s \otimes e) = s(u)\, e$ for all $s \in \Gamma_\infty(H^*)$ and $e \in \Gamma_\infty(E)$. In particular, for each $f \in A^0$ we define

$$u[f] = (\rho(df))(u).$$

In this case, it is clear that

PROPOSITION 2.5. Let (H, ∇_H) be a partial connection for E.

(i) $(u_1 + u_2)(\nabla_H s) = u_1(\nabla_H s) + u_2(\nabla_H s)$ $\quad (u_1, u_2 \in \Gamma_\infty(E), s \in \Gamma_\infty(E))$

(ii) $(fu)(\nabla_H s) = f \cdot u(\nabla_H s)$ $\quad (f \in A^0,\ u \in \Gamma_\infty(H))$

(iii) $u(\nabla_H(s_1 + s_2)) = u(\nabla_H s_1) + u(\nabla_H s_2)$ $\quad (s_1, s_2 \in \Gamma_\infty(E))$

(iv) $u(\nabla_H(fs)) = u[f] \cdot s + (fu)(\nabla_H s)$.

As in the first section, for a complex analytic manifold M with $\dim_C M = m$ we have

$$\tau = T \oplus \bar{T} \quad \text{and} \quad \tau^* = T^* \oplus \bar{T}^*.$$

Let E be a holomorphic vector bundle on M, *i.e.*, for each $x \in M$ there exists an open neighborhood U of x and a homeomorphism

$$h : U\times C^n \longrightarrow \pi^{-1}(U),$$

such that $h|x\times C^n : x\times C^n \longrightarrow \pi^{-1}(x)$ is a complex linear map, where $E=(E, \pi, M)$ and $\dim_c E=n$. We shall denote the space of all holomorphic section of $E|U$ (U is an open subset of M) by $\Gamma(E|U)$.

PROPOSITION 2.6. There is a unique operator

$$\overline{\nabla} : \Gamma_\infty(E) \longrightarrow \Gamma_\infty(\overline{T}^*\otimes E)$$

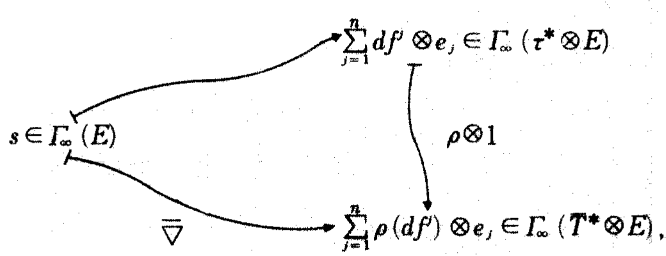
such that

$$\Gamma(E|U) = \mathrm{Ker}\ \{\overline{\nabla} : \Gamma_\infty(E) \longrightarrow \Gamma_\infty(\overline{T}^*\otimes E)\}.$$

PROOF. If $\{e_1, \cdots\cdots, e_n\}$ is a holomorphic frame for E, each element $s\in\Gamma_\infty(E)$ can be uniquely expressed by

$$s=f^1e_1+\cdots\cdots+f^ne_n \quad (f^i\in A^0 \text{ for } i=1,2,\cdots\cdots,n).$$

Then, $\overline{\nabla}$ is defined by the commutative diagram



where ρ is the projection $\tau^* \to \overline{T}^*$ and

$$df^j = \frac{\partial f^j}{\partial z^1}dz^1+\cdots\cdots+\frac{\partial f^j}{\partial z^m}dz^m+\frac{\partial f^j}{\partial \bar{z}^1}d\bar{z}^1+\cdots\cdots+\frac{\partial f^j}{\partial \bar{z}^m}d\bar{z}^m$$

$(j=1,2,\cdots\cdots,n)$. If f^j is holomorphic then s is a holomorphic section and $\frac{\partial f^j}{\partial \bar{z}^k}=0$ for $k=1,2,\cdots\cdots,m$. Since

$$\rho(dz^j)=0 \text{ for } j=1,2,\cdots\cdots,m,$$

if s is holomorphic then $\overline{\nabla}s=0$. Q. E. D.

In particular, since $\overline{\nabla}e_j=0$ for $j=1,2,\cdots\cdots,n$ the matrix of $\overline{\nabla}$ is

$$\begin{pmatrix} 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \end{pmatrix}$$

In fact, $(\overline{T}, \overline{\nabla})$ is a partial connection for E, and the connection ∇ for E which is an extension of $\overline{\nabla}$ has the connection matrix

$$\theta=\begin{pmatrix} \theta_{11}\cdots\cdots\cdots\cdots\theta_{1n} \\ \cdots\cdots\cdots\cdots\cdots \\ \theta_{n1}\cdots\cdots\cdots\cdots\theta_{nn} \end{pmatrix} \qquad \text{(B)}$$

where $\theta_{ij}\in\Gamma(T^*)$ for $1\leqq i,\ j\leqq n$.

DEFINITION 2.7. A holomorphic foliation $\mathcal{F}$ of dimension r on M is given by an open covering $\{U_\alpha\}$ of M and a holomorphic function

$\varphi_\alpha : U_\alpha \longrightarrow C^{m-r}$ satisfying (i) and (ii) :

(i) φ_α is a submersion, and for each $x\in U_\alpha$ $\varphi_\alpha^{-1}(\varphi_\alpha(x))$ is a complex analytic r-dimensional submanifold of U_α.

(ii) If $x\in U_\alpha\cap U_\beta$ then

$$\varphi_\alpha^{-1}(\varphi_\alpha(x))\cap U_\beta=\varphi_\beta^{-1}(\varphi_\beta(x))\cap U_\alpha .$$

The tangent space $T(\varphi_\alpha^{-1}(\varphi_\alpha(x)))_x$ which is the tangent space to the foliation $\mathcal{F}$ at x will be denoted by $T\mathcal{F}_x$, and $T\mathcal{F}$ will represent the r-dimensional holomorphic subbundle of $T(M)=T$ whose fibre over $x\in M$ is $T\mathcal{F}_x$. A holomorphic subbundle F of T is said to be integrable if M has a holomorphic foliation $\mathcal{F}$ such that $F\cong T\mathcal{F}$. This is equivalent to saying that a holomorphic subbundle F of T is integrable if $\Gamma_\infty(F)$ is closed under the bracket operation ([12]).

PROPOSITION 2.8. A holomorphic subbundle F of T is integrable if and only if for each open subset U of M and $r_1, r_2\in\Gamma(F|U)$ $[r_1, r_2]\in\Gamma(F|U)$.

PROOF. By our definition, if F is integrable then it is clear that $[r_1, r_2]\in\Gamma(F|U)$. Conversely, $[r_1, r_2]\in\Gamma(F|U)$ means that a local coordinate $(z^1,\cdots\cdots,z^m)$ for $x\in U$ there exists a base $\{\frac{\partial}{\partial z^{i_1}},\cdots\cdots,\frac{\partial}{\partial z^{i_k}}\,|\,1\leqq i_1<\cdots\cdots<i_k\leqq m\}$ of F_x. Since each $s\in\Gamma_\infty(F|U)$ can be then expressed by

$$s=f^{i_1}\frac{\partial}{\partial z^{i_1}}+\cdots\cdots+f^{i_k}\frac{\partial}{\partial z^{i_k}} \quad (f^{i_1},\cdots\cdots,f^{i_k}\in A^0),$$

it follows that for $s,\ t\in\Gamma_\infty(F|U)$ $[s, t]\in\Gamma_\infty(F|U)$. Q. E. D.

Let us assume that F is an integrable holomorphic subbundle of T. Then $\widetilde{F}=T/F$ is also holomorphic. For the projection $\eta : T\longrightarrow\widetilde{F}$ and $s\in\Gamma_\infty(\widetilde{F})$ there exists an element $\tilde{s}\in\Gamma_\infty(T)$ such that $\eta(\tilde{s})=s$. If we put

$$\eta[u, \tilde{s}] = \langle u, s\rangle \text{ for } u \in \Gamma_\infty(F)$$

$$\eta[u, \tilde{s}] = \langle u, s\rangle \begin{cases} = 0 & \text{if } \tilde{s} \in \Gamma_\infty(F) \\ \neq 0 & \text{if } \tilde{s} \notin \Gamma_\infty(F) \end{cases}$$

It follows from the properties of the bracket product that

(i) $\langle u_1+u_2, s\rangle = \langle u_1, s\rangle + \langle u_2, s\rangle$, $\langle u, s_1+s_2\rangle = \langle u, s_1\rangle + \langle u, s_2\rangle$

(ii) $\langle fu, s\rangle = f\langle u, s\rangle$, $\langle u, fs\rangle = u[f]s + f\langle u, s\rangle$,

where $u, u_1, u_2, \in \Gamma_\infty(F)$, $s, s_1, s_2 \in \Gamma_\infty(\widetilde{F})$ and $f \in A^0$.

We shall define a partial connection $\nabla_{F\oplus\bar{T}} : \Gamma_\infty(\widetilde{F}) \longrightarrow \Gamma_\infty((F^*\oplus \bar{T}^*)\otimes\widetilde{F})$ as follows. Since $\Gamma_\infty((F^*\oplus \bar{T}^*)\otimes\widetilde{F}) = \Gamma_\infty(F^*\otimes\widetilde{F}) + (\bar{T}^*\otimes\widetilde{F})$ we put

$$\nabla_{F\oplus\bar{T}} = (\nabla_F, \nabla_{\bar{T}}),$$

where (F, ∇_F) and $(\bar{T}, \nabla_{\bar{T}})$ are parital connections for $\widetilde{F}$. $\nabla_F : \Gamma_\infty(\widetilde{F}) \to \Gamma_\infty(\widetilde{F})$ is defined by $u(\nabla_F s) = \eta[u, \tilde{s}] = \langle u, s\rangle$ for all $u \in \Gamma_\infty(F)$ and $s \in \Gamma_\infty(\widetilde{F})$. Then, it is easily proved that

(i) $u(\nabla_F fs) = \langle u, fs\rangle = u[f]s + f\langle u, s\rangle$

(ii) $(fu)\nabla_F s = \langle fu, s\rangle = f\langle u, s\rangle$ for $u \in \Gamma_\infty(F)$, $s \in \Gamma_\infty(\widetilde{F})$ and $f \in A^0$.

Next, we put $\nabla_{\bar{T}} = \bar{\nabla}$ which is the same one as in Proposition 2.6.

Then, for $u \in \Gamma_\infty(F)$, $v \in \Gamma_\infty(\bar{T})$ and $s \in \Gamma_\infty(\widetilde{F})$ we have

$$u(\nabla_{F\oplus\bar{T}} s) = \langle u, s\rangle \text{ and } v(\nabla_{F\oplus\bar{T}} s) = v(\bar{\nabla} s).$$

An extension ∇ of $\nabla_{F\oplus\bar{T}}$ is called a basic connection for $\widetilde{F}$. By (B) above every entry of the connection matrix of ∇ is an element of $\Gamma(T^*)$.

PROPOSITION 2.9. Under the above situation we set $k = \dim_C F_x (x \in M)$, ∇ = a basic connection for $\widetilde{F}$ and $K = K(\nabla)$. If φ is an invariant homogeneous polynomial of degree l and $m-k < l \leqq m$ then $\varphi(K) = 0$.

PROOF. Let U be an open neighborhood of $x \in M$ with a complex analytic coordinate system $(z^1, \cdots, z^m)$. Then $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^m}\}$ is a base of $\Gamma(E|U)$. Since the set of all smooth complex-valued differential forms $A(U)$ on U is a ring with the usual addition and wedge product of differential forms, we define the ideal $I(F, U)$, which is generated by $dz^{k+1}, \cdots, dz^m$. Then, for $\omega, \omega_1, \cdots, \omega_{m-k+1} \in I(F, U)$, it follows that $d\omega \in I(F, U)$ and $\omega_1 \wedge \cdots \wedge \omega_{m-k+1} = 0$. Let $\theta = \|\theta_{ij}\|$ be the connection matrix of ∇ with respect to the frame $\eta\left(\frac{\partial}{\partial z^{k+1}}\right), \cdots, \eta\left(\frac{\partial}{\partial z^m}\right)$, where $\eta : T \longrightarrow \widetilde{F}$ is the projection as above. Since

$$\left[\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right]=0 \qquad 1\leq j,\ k\leq m$$

and for $u\in\Gamma_\infty(F)$, $\gamma\in\Gamma_\infty(T)$, $u(\nabla(\eta\gamma))=\eta[u,\gamma]$ we have for each θ_U

$$\frac{\partial}{\partial z^1}(\theta_U)=\cdots\cdots=\frac{\partial}{\partial z^k}(\theta_U)=0,$$

and thus $\theta_U\in I(F,U)$. Hence, for the curvature matrix $K=\|K_U\|$ of ∇ with respect to $\eta\left(\frac{\partial}{\partial z^{k+1}}\right),\cdots,\eta\left(\frac{\partial}{\partial z^m}\right)$, each K_U is in $I(F,U)$. In this case, if we put $K(\nabla)=K$ then on U

$$\varphi(K)\mid U=\widetilde{\varphi}\ (\sigma_1(K),\cdots,\sigma_l(K))$$

(see the statements before Definition 2.3). Since $l\geq m-k+1$ $\varphi(K)$ vanishes on U.

Q. E. D.

DEFINITION 2.10. Let $\mathcal{O}(M)$ be the category consisting of all open subsets of M and all inclusion maps, let

$$\begin{array}{ccc} \hat{\mathcal{O}}:\mathcal{O}(M) & \longrightarrow & A_b \\ \Psi & & \Psi \\ U & \rightsquigarrow & \hat{\mathcal{O}}(U)=\{f: U\longrightarrow C \mid f:\text{holomorphic function}\} \end{array}$$

be the cofunctor defined by $\hat{\mathcal{O}}(i)(f)=f\mid V$, where $i: V\hookrightarrow U$ in $\mathcal{O}(M)$ and A_b the category of all abelian groups. Then, $\hat{\mathcal{O}}$ is a presheaf. Let $\mathcal{O}$ be the sheaf induced from $\hat{\mathcal{O}}$. Then $\mathcal{O}$ is called the structure sheaf of M. In consequence, $\mathcal{O}$ is the sheaf of germs of holomorphic functions from open subsets of M to C.

Let T be the holomorphic tangent bundle of M as before, L be a holomorphic line bundle over M, and $\eta: L\rightarrow T$ be a holomorphic vector bundle map. Given an isolated zero $x\in M$ of η, choose a nonvanishing holomorphic section X_x of L. Also about x, we choose a complex analytic coordinate system $z^1,\cdots,z^m$ with origin at x.

Then we can write as

$$\eta(X_x)=\sum_{i=1}^{m}a_i\frac{\partial}{\partial z^i}=0, \qquad \text{(C)}$$

where a_i are holomorphic functions near x. Let us use the notation that whenever E is a holomorphic vector bundle, $\underline{E}$ shall denote the sheaf of germs of holomorphic sections of E. Then

$$\underline{\eta}:\underline{L}\longrightarrow\underline{T}$$

is injective. Set $\xi=\eta(\underline{L})$ and $Q=\underline{T}/\xi$, then each $y\in M$ ξ_y and Q_y are $\mathcal{O}_y$-modules. We say that ξ is integrable subsheaf of $\underline{T}$ if for each $y\in M$ the stalk ξ_y is closed under the bracket operation for vector fields. Let us assume that ξ is integrable. Then, by (C) above, Q_x is not a free $\mathcal{O}_x$-module. We put

$$\text{Zero}(\eta)=\{x\in M \mid Q_x \text{ is not a free } \mathcal{O}_x\text{-module}\},$$

then we have an one-dimensional foliation on M-Zero(η). On M, however, we have a foliation with singularities.

DEFINITION 2.11. A subsheaf $\xi\subset\underline{T}$ is integrable if

(i) it is coherent

(ii) for each $x\in M$ ξ_x is closed under the bracket operation for vector fields.

Put $Q=\underline{T}/\xi$ and $S=\{x\in M \mid Q_x$ is not a free $\mathcal{O}_x$-module$\}$. S is called the singular set of the foliation which is determined by ξ. In this case, S is a Closed holomorphic submanifold of M, and there is a unique holomorphic subbundle F of $T\mid M\text{-}S$ such that

$$\underline{F}=\xi\mid M-S$$

([3]), and $\dim_C F_x$ is called the leaf dimension of ξ.

3. Foliations with Singularities

As in the second section, we assume that M is a complex analytic manifold with $\dim_C(M)=m$ $(m\geq 2)$. Therefore M_R is a real 2m-dimensional smooth manifold.

Let $H^*(M;\ C)$ be the de Rham cohomology ring of M. We put

$$H^j_{comp}(M;C)=\varinjlim H^j(M,M-K;C),$$

where K varies over the directed set consisting of all compact subsets of M. Then the Poincarè dulity theorem says that

$$H^j_{comp}(M;\ C)\cong H_{2m-j}(M;\ C),$$

because of that

(i) M is orientable

(ii) $H_*(M;\ C)$ is obtained by the dual of $H^*(M;\ C)$ (for example, by Currents) ([1], [11] and [13]).

Hence, for an open subset $U(\subset M)^r$ and a compact Z which is a deformation retract of U, we have the isomorphisms

$$H^{j}_{comp}(U;\ C) \xrightarrow{\cong} H_{2m-j}(U;\ C) \xleftarrow{\cong} H_{2m-j}(Z;\ C). \qquad \text{(D)}$$

Let E be smooth complex vector bundle over M. If we denote the total Chern class of E in $H^*(M;C)$ as

$$c(E)=1+c_1(E)+\cdots\cdots+c_m(E),$$

then it is invertible in $H^*(M;\ C)$.

DEFINITION 3.1. For two smooth complex vector bundles E_0 and E_1 over M the total Chern class of the virtual bundle E_0-E_1 is defined by

$$\begin{aligned} c(E_0-E_1) &= c(E_0)/c(E_1) \\ &= 1+c_1(E_0-E_1)+\cdots\cdots+c_m(E_0-E_1), \end{aligned}$$

where $c_j(E_0-E_1)\in H^{2j}(M;\ C)$. More generally, let $E_q,\ E_{q-1},\cdots,E_0$ be smooth complex vector bundles over M and put $\varepsilon(i)=(-1)^i$. Then the total Chern class of the virtual bundle $\sum_{i=0}^{q}(-1)^iE_i=\zeta$ is defined by

$$\begin{aligned} c\left(\sum_{i=0}^{q}(-1)^iE_i\right) &= \prod_{i=0}^{q}(c(E_i))^{\varepsilon(i)} \\ &= 1+c_1(\zeta)+\cdots\cdots+c_m(\zeta), \end{aligned}$$

where $c_j(\zeta)\in H^{2j}(M;\ C)$.

As in the second section, if we define

(i) $\varphi=\widetilde{\varphi}(\sigma_1,\cdots,\sigma_l)$ is an invariant homogeneous polynomial of degree l,

(ii) $\nabla_q,\cdots,\nabla_0$ are connections for $E_q,\cdots,E_0$ respectively,

(iii) $\sigma_j(K_q|K_{q-1}|\cdots\cdots|K_0)$ by

$\sigma_j(K_q|K_{q-1}|\cdots\cdots|K_0)\in\Gamma_\infty(\wedge^{2j}\tau^*)$ $(j=1,\cdots,n)$ and

$\prod_{i=0}^{q}(\|1+K_i\|)^{\varepsilon(i)}=1+\sigma_1(K_q|\cdots\cdots|K_0)+\cdots+\sigma_n(K_q|K_{q-1}|\cdots\cdots|K_0)$,

where $K_i(\nabla_i)=K_i$, then we have the following:

(i)′ $\varphi(\zeta)=\widetilde{\varphi}(c_1(\zeta),\cdots,c_l(\zeta))\in H^{2l}(M;\ C)$

(ii)′ $d\sigma_j(K_q|\cdots|K_0)=0$ and

$$[\sigma_j(K_q|\cdots|K_0)]=(2\pi/\sqrt{-1})^j\ c_j(\zeta)$$

(iii)′ $\omega_j=\sigma_j(K_q|\cdots|K_0)$ and $\varphi(K_q|\cdots|K_0)=\widetilde{\varphi}(\omega_1,\cdots,\omega_l)$

$$\Longrightarrow d\varphi(K_q|\cdots|K_0)=0 \text{ and } [\varphi(K_q|\cdots|K_0)]=(2\pi/\sqrt{-1})^l\varphi(\zeta). \qquad \text{(E)}$$

DEFINITION 3.2. For an exact sequence of smooth complex vector bundles over M;

$$0\longrightarrow E_q\xrightarrow{\eta_q}E_{q-1}\longrightarrow\cdots\longrightarrow E_0\xrightarrow{\eta_0}E_{-1}\longrightarrow 0,$$

let $\nabla_q, \cdots, \nabla_0, \nabla_{-1}$ be connections for $E_q, \cdots, E_0, E_{-1}$, respectively. Then $(\nabla_q, \cdots, \nabla_{-1})$ is <u>compatible</u> with the exact sequence if the diagram

$$\begin{array}{ccc} \Gamma_\infty(E_i) & \xrightarrow{\quad\nabla_i\quad} & \Gamma_\infty(\tau^*\otimes E_i) \\ \eta_i \downarrow & & \downarrow 1\otimes\eta_i \\ \Gamma_\infty(E_{i-1}) & \xrightarrow{\quad\nabla_{i-1}\quad} & \Gamma_\infty(\tau^*\otimes E_{i-1}) \end{array}$$

is commutative for $i=q, q-1, \cdots, 1, 0$.

In the above notations, if we give a connection ∇_{-1} for E_{-1}, then there always exist compatible connections $\nabla_q, \cdots, \nabla_0, \nabla_{-1}$ for $E_q, \cdots, E_0, E_{-1}$, respectively.

Let ξ be an integrable subsheaf of $\underline{T}$, k $(1\leq k<m)$ the leaf dimension of ξ and S the singular set. As in the second section there exists a unique holomorphic subbundle F of T such that

$$\underline{F}=\xi\,|M-S.$$

We set $\nu=T\,|M-S/F$.

DEFINITION 3.3. For a connected component Z of S, a Z-sequence β is a triple $(U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$ satisfying the following conditions:

(i) U is an open subset of M and $Z=U\cap S$ is a deformation retract of U,

(ii) $0 \longrightarrow E_q\,|U-Z \xrightarrow{\eta_q} E_{q-1}\,|U-Z \xrightarrow{\eta_{q-1}} \cdots \longrightarrow E_0\,|U-Z \xrightarrow{\eta_0} \nu\,|U-Z \longrightarrow 0$ is an exact sequence of smooth complex vector bundles over U and smooth bundle maps.

DEFINITION 3.4. For a Z-sequence $\beta=(U, (E_q, \cdots, E_0), (\eta_q, \cdots, \eta_0))$ and connections $\nabla_q, \cdots\cdots\nabla_0$ for $E_q, \cdots, E_0$, respectively, $(\nabla_q, \cdots, \nabla_0, \nabla_{-1})$ is <u>fitted to</u> β if

(i) ∇_{-1} is a basic connection for $\nu\,|U-Z$

(ii) there exists a compact subset Σ of U such that

(a) Z is contained in the interior of Σ

(b) $(\nabla_q, \cdots, \nabla_0, \nabla_{-1})$ is compatiable with the exact sequence,

$$0\longrightarrow E_q\,|U-\Sigma \xrightarrow{\eta_q|U-\Sigma} E_{q-1}\,|U-\Sigma \longrightarrow\cdots\cdots\longrightarrow E_0\,|U-\Sigma \xrightarrow{\eta_0|U-\Sigma} \nu\,|U-\Sigma\longrightarrow 0.$$

Under the above notations, we assume that $(\nabla_q, \cdots, \nabla_0, \nabla_{-1})$ is fitted to a Z-sequence $\beta=(U, (E_q, \cdots, E_0), (\eta_q, \cdots, \eta_0))$.

Set $K_i=K(\nabla_i)$ and $\varphi=$an invariant homogeneous polynomial of degree l $(m-k<l\leq m)$. Then, by (D) and (E) above there exists the element $(\xi, Z, \beta)_\varphi\in H_{2m-2l}(Z; C)$ which is determined by $(\sqrt{-1}/2\pi)^l\ \varphi(K_q\,|\cdots|K_0)=\varphi(\zeta)\in H^{2l}(U; C)$, where ζ

$=\sum_{i=0}^{q}(-1)^i E_i.$

THEOREM 3.5. $(\xi, Z, \beta)_\varphi$ depends only on ξ, Z, β and φ, *i.e.*, $(\xi, Z, \beta)_\varphi$ does not depend on the choice of $\nabla_q, \cdots, \nabla_0, \nabla_{-1}$.

In order to prove our theorem we need several lemmas.

LEMMA 3.6. Let $(\nabla_q, \cdots, \nabla_0, \nabla_{-1})$ be compatible with the exact sequence $0 \to E_q \to \cdots \to E_0 \to E_{-1} \to 0$ of smooth complex vector bundle over M. If we put $K_i = K(\nabla_i)$ then $\varphi(K_{-1}) = \varphi(K_q | \cdots | K_0)$, where φ is an invariant homogeneous polynomial of degree $l \leqq m$.

PROOF. It suffices to prove that

$$\|1+K_{-1}\| = \pi_{i=0}^{q} \|1+K_i\|^{\varepsilon(i)}, \quad \varepsilon(i) = (-1)^i.$$

If $q=0$ then $0 \to E_0 \xrightarrow{=} E_{-1} \to 0$ is exact, and thus our assertion is clear. We assume that our lemma is valid for $q-1$, and consider an exact sequence $0 \to E_q \xrightarrow{\eta_q} E_{q-1} \to \cdots \to E_0 \xrightarrow{\eta_0} E_{-1} \to 0$. Then, there exists a smooth subbundle D of E_{q-1} such that $E_{q-1} = D \oplus \eta_q(E_q)$.

By the projection $\rho : E_{q-1} \longrightarrow D$ we have

$$1 \otimes \rho : \tau^* \otimes E_{q-1} \longrightarrow \tau^* \otimes D,$$

and define a connection ∇ for D by $\nabla = (1 \otimes \rho)\nabla_{q-1}$. Then, since $(\nabla, \nabla_{q-2}, \cdots, \nabla_0, \nabla_{-1})$ is compatible with the exact sequence

$$0 \longrightarrow D \longrightarrow E_{q-2} \longrightarrow \cdots\cdots \longrightarrow E_0 \longrightarrow E_{-1} \longrightarrow 0$$

Hence, by our assumption we have

$$\|1+K_{-1}\| = \|1+K(\nabla)\|^{\varepsilon(q-1)} \pi_{i=1}^{q-2} \|1+K_i\|^{\varepsilon i}.$$

Since $0 \to E_q \to E_{q-1} \to D \to 0$ .. exact and $(\nabla_q, \nabla_{q-1}, \nabla)$ is compatible with this exact sequence we have

$$\|1+K(\nabla)\| = \|1+\nabla_{q-1}\| \cdot \|1+\nabla_q\|^{-1}$$

This completes our proof. Q. E. D.

LEMMA 3.7. For a closed subset B of M and a smooth complex vector bundle E over M, let ∇ be a connection for $E|M-B$. If Σ is a closed subset of M containing B in its interior, then there exists a connection $\tilde{\nabla}$ for E such that ∇ coincides with $\tilde{\nabla}$ on $E|M-\Sigma$.

PROOF. Since M is locally compact and Hausdorff there exists a smooth function $\psi: M \to \mathbf{R}$ such that

(a) it vanishes on a neighborhood of B and

(b) $\psi = 1$ on $M - \Sigma$.

Take a connection D for E and set $\widetilde{\nabla} = \psi\nabla + (1-\psi)D$. Then the required connection is $\widetilde{\nabla}$. Q. E. D.

Let M and N be complex analytic manifolds, E be a smooth vector bundle over N, and $g: M \to N$ be a smooth map. The $g^!(E)$ is the pullback bundle over M such that for each $x \in M$

$$g^!(E)_x = E_{gx}.$$

For a connection ∇ for E the pull-back connection $g^!(\nabla)$ for $g^!(E)$ is defined as follows. Let $e_1, \cdots, e_r$ be a smooth frame of E and $\theta = \| \theta_{ij} \|$ be the connection matirx ∇ with respect to $e_1, \cdots, e_r$ for U (open in N). If the connection matrix of $g^!(\nabla)$ with respect to $g^!(e_1), \cdots, g^!(e_r)$ is $\omega = \|\omega_{ij}\|$ then $\omega_{ij} = g^* \theta_{ij}$, where

$$g^*: \Gamma_\infty(\tau_N^* | U) \longrightarrow \Gamma_\infty(\tau^*_M | g^{-1}(U))$$

is defined from g. If $\eta: E' \to E$ is a map of smooth vector bundles over N, then $g^!(\eta): g^!(E') \to g^!(E)$.

LEMMA 3.8. Under the notations in Definition 3.3, let W be an open subset of $M - S$. Assume that

(i) ∇ and ∇' are two basic connections for $\nu | W$

(ii) $\widetilde{W} = W \times [0, 1]$ and $\rho: \widetilde{W} \to W$, $t: \widetilde{W} \to [0, 1]$ are the projections

(iii) $D = t\rho^!(\nabla') + (1-t)\rho^!(\nabla)$ is a connection for $\rho^!(\nu | W)$

(iv) $K = K(D)$ and φ is an invariant homogeneous polynomial of degree l such that $m-k < l \leq m$.

Then we have $\varphi(K) = 0$.

PROOF. The proof is very much like the proof of Proposition 2.9. For a chart $\{W_\alpha, \varphi_\alpha\}$ of $x \in W$ $(W_\alpha \subset W)$, let its complex analytic coordinate system be $\{z^1, \cdots, z^m\}$, and $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^k}\}$ be a base of $\Gamma(F | W_\alpha)$. For $\widetilde{W}_\alpha = W_\alpha \times [0, 1]$, let $A(\widetilde{W}_\alpha)$ be the ring of all smooth complex valued differential forms over $\widetilde{W}_\alpha$. If $I(F, \widetilde{W}_\alpha)$ is the ideal of $A(\widetilde{W}_\alpha)$ generated by $\rho^*(dz^{k+1}), \cdots, \rho^*(dz^m)$ then the following hold

(a) $\forall \omega \in I(F, \widetilde{W}_\alpha)$ $d\omega \in I(F, \widetilde{W}_\alpha)$

(b) $\omega_1, \cdots, \omega_{m-k+1} \in I(F, \widetilde{W}_\alpha) \Rightarrow \omega_1 \wedge \cdots \wedge \omega_{m-k+1} = 0.$

Let $\theta = \|\theta_{ij}\|$ and $\theta' = \|\theta'_{ij}\|$ be the connection matrices of ∇ and ∇', respectively, with respect to $\eta\left(\frac{\partial}{\partial z^{k+1}}\right), \cdots, \eta\left(\frac{\partial}{\partial z^m}\right)$, where $\eta : T \to \nu$ is the projection.

If $\omega = \|\omega_{ij}\|$ is the connection matrix of D with respect to $\rho'\left(\eta\left(\frac{\partial}{\partial z^{k+1}}\right)\right), \cdots, \rho\left(\eta\left(\frac{\partial}{\partial z^m}\right)\right)$, then by the above statements

$$\omega_{ij} = t\rho^*(\theta'_{ij}) + (1-t)\rho^*(\theta_{ij}) \in I(F, \widetilde{W}_\alpha),$$

because of that ∇ and ∇' are basic. As in the proof of Proposition 2, 9, then $K_{ij} \in I(F, \widetilde{W}_\alpha)$, where $K = \|K_{ij}\|$ is the curvature matrix of D. When $\varphi = \widetilde{\varphi}(\sigma_1, \cdots, \sigma_l)$ we get

$$\varphi(K) \mid \widetilde{W}_\alpha = 0$$

from the proof of Proposition 2. 9. Q. E. D.

LEMMA 3.9. Given a Z-sequence $\beta = (U, (E_q, \cdots E_0,), (\eta_q, \cdots, \eta_0))$ and a basic connection ∇_{-1} for $\nu \mid U-Z$, then there exist connections $\nabla_q, \cdots, \nabla_0$ for $E_q, \cdots, E_0$, respectively, such that $(\nabla_q, \cdots, \nabla_0, \nabla_{-1})$ is fitted to β.

PROOF. As is well known, there exist connections $D_q, \cdots, D_0$, for $E_q|U-Z, \cdots, E_0|U-\Sigma$ respectively, such that $(D_q, \cdots, D_0, \nabla_{-1})$ is compatible with the exact sequence

$$0 \to E_q \mid U-Z \xrightarrow{\eta_q} E_{q-1} \mid U-Z \xrightarrow{\eta_{q-1}} \cdots \to E_0 \mid U-Z \xrightarrow{\eta_0} \nu \mid U-Z \to 0.$$

Choose a compact subset Σ of U containing Z in its interior. Then, by Lemma 3.7, there exist connections $\nabla_q, \cdots, \nabla_0$ for $E_q, \cdots, E_0$ such that for $i = q, q-1, \cdots, 0$ ∇_i on $E_i \mid U-\Sigma$ coincides with D_i. In this case, our required connections are ∇_q, $\nabla_{q-1}, \cdots, \nabla_0$. Q. E. D.

PROOF of THEOREM 3.5. We have to prove that

(i) $\varphi(K_q \mid \cdots \mid K_0)$ has compact support

(ii) for an another $(\nabla'_q, \cdots, \nabla'_{-1})$ which is fitted to β, there exists a $(2l-1)$-form ω such that

(a) ω has compact support

(b) $d\omega = \varphi(K'_q \mid \cdots \mid K'_0) - \varphi(K_q \mid \cdots \mid K_0)$,

where $K(\nabla'_i) = K'$ for $i = q, q-1, \cdots, 1$.

By Lemma 3.6, we immediately have

$$\varphi(K_q|\cdots|K_0)\,|\,U-\Sigma=\varphi(K_{-1})\,|\,U-\Sigma.$$

Since $\nabla_{-1}\,|U-\Sigma$ is basic $0=\varphi(K_{-1})|U-\Sigma=\varphi(K_q|\cdots|K_0)\,|U-\Sigma$ by Proposition 2. 9. This is a proof of (i).

Define $\widetilde{U}$, $\widetilde{\Sigma}$ and $\widetilde{Z}$ by

$$\widetilde{U}=U\times[0,1],\quad \widetilde{Z}=Z\times[0,1] \text{ and } \widetilde{\Sigma}=\Sigma\times[0,1].$$

Taking the projections $\rho:\widetilde{U}\to U$ and $t:\widetilde{U}\to[0,1]$, we have the pull-back bundle $\rho^!(E_i)$, the pull-back connections $\rho^!(\nabla_i)$ and $\rho^!(\nabla_i')$ for $i=q,\ q-1,\cdots,0$. Define

$$D_i=t\ \rho^!(\nabla_i')+(1-t)\ \rho^!(\nabla i)\quad (i=0,\cdots,q)\quad (\text{on } \widetilde{U}),$$

and on $\widetilde{U}-\widetilde{Z}$

$$D_{-1}=t\rho^!(\nabla_{-1}')+(1-t)\rho^!(\nabla_{-1}).$$

Set $\widetilde{K}_i=K(D_i)$ $(i=q,\cdots,0)$ and $\widetilde{K}_{-1}=K(D_{-1})$. If we define $i_0:U\to\widetilde{U}$ and $i_1:U\to\widetilde{U}$ by $i_0(u)=(u,0)$ and $i_1(u)=(u,1)$ for $u\in U$, respectively, then we have

$$i_0{}^*\varphi(\widetilde{K}_q|\cdots|\widetilde{K}_0)=\varphi(K_q|K_{q-1}|\cdots|K_0)$$

$$i_1{}^*\varphi(\widetilde{K}_q|\cdots|\widetilde{K}_0)=\varphi(K_q'|K_{q-1}'|\cdots|K_0').$$

By (E) above, there exists a $(2l-1)$-form $\widetilde{\omega}$ on $\widetilde{U}$ such that

$$\varphi(\widetilde{K}_q|\cdots|\widetilde{K}_0)=(2\pi/\sqrt{-1})^l\varphi(\widetilde{\zeta})+d\widetilde{\omega},$$

where $\widetilde{\zeta}=\sum_{i=0}^{q}(-1)^i\ \rho^!(E_i)$. Since

$$\varphi(\widetilde{K}_q|\cdots|\widetilde{K}_0)|\widetilde{U}-\widetilde{\Sigma}=\varphi(\widetilde{K}_{-1})|\widetilde{U}-\widetilde{\Sigma}$$

and $D_{-1}\,|\,\widetilde{U}-\widetilde{\Sigma}$ is basic,

$$\varphi(\widetilde{K}_q|\cdots|\widetilde{K}_0)\,|\,\widetilde{U}-\widetilde{\Sigma}=\varphi(\widetilde{K}_{-1})\,|\,\widetilde{U}-\widetilde{\Sigma}=0.$$

This implies that $d\widetilde{\omega}\,|\,\widetilde{U}-\widetilde{\Sigma}=0$, and thus $\widetilde{\omega}$ has compact support. On the other hand,

$$\begin{aligned}&\varphi(K_q'|\cdots|K_0')-\varphi(K_q|\cdots|K_0)\\&=i_1{}^*\varphi(\widetilde{K}_q|\cdots|\widetilde{K}_0)-i_0{}^*\varphi(\widetilde{K}_q|\cdots|\widetilde{K}_0)\\&=(2\pi/\sqrt{-1})^l\varphi(\zeta)+di_1{}^*\widetilde{\omega}-(2\pi/\sqrt{-1})^l\varphi(\zeta)-di_0{}^*\widetilde{\omega}\\&=d(i_1{}^*\widetilde{\omega}-i_0{}^*\widetilde{\omega}).\end{aligned}$$

So we set $\omega=i_1{}^*\widetilde{\omega}-i_0{}^*\widetilde{\omega}$ then ω is a $(2l-1)$-form on U which has compact support, and

$$d\omega=\varphi(K_q'|\cdots|K_0')-\varphi(K_q|\cdots|K_0)$$

Therefore $(2\pi/\sqrt{-1})^l\varphi(\zeta)=[\varphi(K_q|\cdots|K_0)]=[\varphi(K_q'|\cdots|K_0')]$, and thus $(\xi,Z,\beta)_\varphi$ does not depend on the choice of $\nabla_q,\cdots,\nabla_{-1}$. Q. E. D.

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