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PSEUDO-UNIVERSAL SPACES OF VECTOR BUNDLES

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§ 1. Introduction

In [1], Bott studied on foliations by means of connections and the Pontryagin classes. Such study is important in the theory of vector bundles.

Motivated by [1], we introduce the concept of pseudo-universal spaces of vector bundles. Our main result is that the set $I_{so}(\mathscr{C}(M))$ of certain isomorphism classes of some continuous functors related to the category $\mathscr{C}(M)$ of all differentiable manifolds and all manifold homomorphisms is a pseudo-universal space of $\mathscr{C}(M)$ itself.

Section 2 is devoted to give some preliminary facts on vector bundles and connections.

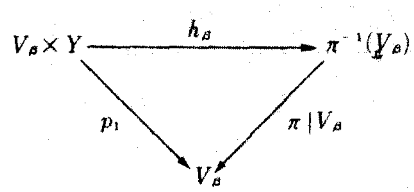
In Section 3, we derive the concept and some illustrations of topological categories. In order to describe the topological category Γ_q in Example 3.12, we obtain some necessary facts on sheaves and Γ_q-cocycles. In particular, a relation between Γ_q-cocycles and connections is given in Proposition 3.11.

In Section 4, we define the continuities of functors between topological categories and of natural transformations of such functors, and pseudo-universal spaces of categories of topological spaces. Our main result is Theorem 4.8.

§ 2. Vector Bundles and Connections

Let G be an effective topological transformation group of a topological space Y. For topological spaces E and X, a coordinate bundle $\mathscr{B}=(E, \pi, X, Y, G)$ satisfies the following conditions, where $\pi : E\to X$ is a continuous surjection:

(i) There exists an open covering $\{V_\beta\}_{\beta\in J}$ of X such that, for each $\beta\in J$, the following diagram commutes



where h_β is a homeomorphism and $p_1(x, y)=x$ for each $(x, y)\in V_\beta\times Y$. In this case, h_β is called the *coordinate function.*

(ii) If the function $h_{\beta,x}: Y\to\pi^{-1}(x)$ is defined by $h_{\beta,x}(y)=h_\beta(x, y)$, then for all $(\alpha, \beta)\in J\times J$ and $x\in V_\alpha\cap V_\beta$,

$$h_{\beta,x}^{-1}\circ h_{\alpha,x}: Y \to Y$$

is a homeomorphism. Since G is isomorphic to the set of all homeomorphisms of Y to itself, we see that $h_{\beta,x}^{-1}\circ h_{\alpha,x}=g_{\beta\alpha}(x)\in G$. That is,

$$g_{\beta\alpha}: V_\alpha\cap V_\beta \to G$$

is continuous. If we put $Y_x=\pi^{-1}(x)$ for each $x\in X$, Y_x is called the *fiber* at x.

The following proposition is easily proved ([13]).

Proposition 2.1 (i) $h_\beta^{-1}\circ h_\alpha(x, y)=(x, g_{\beta\alpha}(x)y)$ for every $(x, y)\in(V_\alpha\cap V_\beta)\times Y$.

(ii) For $x\in V_\alpha$, $g_{\alpha\alpha}(x)=$ the identity of G.

(iii) For $x\in V_\alpha\cap V_\beta\cap V_\gamma$, $g_{\gamma\beta}(x)\circ g_{\beta\alpha}(x)=g_{\gamma\alpha}(x)$.

(iv) For $x\in V_\alpha\cap V_\beta$, $(g_{\beta\alpha}(x))^{-1}=g_{\alpha\beta}(x)$.

(v) If $p_\beta: \pi^{-1}(V_\beta)\to Y$ is defined by $p_\beta(b)=h_{\beta,x}^{-1}(b)$, $\pi(b)=x$, then

$$p_\beta\circ h_\beta(x, y)=y, \qquad h_\beta(\pi(b), p_\beta(b))=b$$

$$g_{\beta\alpha}(\pi(b)\cdot p_\alpha(b))=p_\beta(b), \qquad \pi(b)\in V_\alpha\cap V_\beta$$

Definition 2.2 In a *coordinate bundle* $\mathscr{B}=(E, \pi, X, Y, G)$ as defined above, E is the *total space* of $\mathscr{B}$, X is the *base space* of $\mathscr{B}$, G is the *structure group* of $\mathscr{B}$, Y is a *fiber space*, (V_β, h_β) is a *chart* of $\mathscr{B}$ for each $\beta\in J$ and $g_{\alpha\beta}$ is a *coordinate transformation* of $\mathscr{B}$ where $(\alpha, \beta)\in J\times J$.

A coordinate bundle $\mathscr{B}=(E, \pi, X, Y, G)$ with charts $\{(V_\beta, h_\beta)\}_{\beta\in J}$ and coordinate transformations $\{g_{\alpha\beta}\}_{\alpha,\beta\in J}$ is *equivalent* to $\mathscr{B}$ with charts $\{(V'_\gamma, h'_\gamma)\}_{\gamma\in J'}$ and coordinate transformations $\{g'_{\gamma\iota}\}_{\gamma,\iota\in J'}$, if there exist continuous maps

$$\bar{g}_{\gamma\alpha}: V_\beta\times V'_\gamma \to G, \qquad \beta\in J,\ \gamma\in J'$$

such that

$$\bar{g}_{\gamma\alpha}(x)=\bar{g}_{\gamma\beta}(x)\circ g_{\beta\alpha}(x), \quad x\in V_\alpha\cap V_\beta\cap V'_\gamma;$$

$$\bar{g}_{\iota\beta}(x)=g'_{\iota\gamma}(x)\circ\bar{g}_{\gamma\beta}(x), \quad x\in V_\beta\cap V'_\gamma\cap V'_\iota.$$

A *fiber bundle* is defined to be an equivalence class of coordinate bundles. Let $\mathscr{B}=(E, \pi, B, Y, G)$ be a fiber bundle. If $Y=R^n$ ($R=$reals) and $G=GL(n, R)$, then $\mathscr{B}=(E, \pi, B, R^n, GL(n, R))$ is called a *real vector bundle over B with dimension n.* In general, an n-dimensional real vector bundle $\mathscr{B}=(E, \pi, B, R^n, GL(n, R))$ is written as $\mathscr{B}^n=(E, \pi, B)$ or $(E(\mathscr{B}^n), \pi, B(\mathscr{B}^n))$.

Let $Vect_k(B)$ be the set of all isomorphism classes of k-dimensional vector bundles over a topological space B and PS be the category of all paracompact spaces and homotopy classes of continuous maps. Then

$$Vect_k : PS \longrightarrow E_{ns}$$

is a cofunctor, where E_{ns} is the category of all sets and functions. That is, for $[f] : B \to B'$ in PS,

$$\begin{array}{ccc} Vect_k([f]) : Vect_k(B') & \longrightarrow & Vect_k(B) \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ \xi'^k & \rightsquigarrow & \{f^*(\xi'^k)\}, \end{array}$$

where $[f]$ is the homotopy class containing f and $\{f^*(\xi'^k)\}$ is the isomorphism class containing $f^*(\xi'^k)$.

For the k-dimensional Grassmann variety $G_k(R^n)$ $(n \geqq k)$, let γ_n^k be the *canonical k-dimensional real vector bundle* over $G_k(R^n)$. Then γ_n^1 is the *canonical line bundle* over $RP^{n-1} = G_1(R^n)$, where RP^{n-1} is the $(n-1)$-dimensional real projective space. In particular, γ^k denotes the canonical k-dimensional real vector bundle over $G_k(R^\infty)$.

As is well known, for $B \in PS$, each $\xi^k \in Vect_k(B)$ is isomorphic to $f^*(\gamma^k)$ for some $f : B \to G_k(R^\infty)$. Let $[B, G_k(R^\infty)]$ be the set of all homotopy classes of continuous functions from B to $G_k(R^\infty)$. Define

$$\varphi_B : [B, G_k(R^\infty)] \longrightarrow Vect_k(B)$$

by taking $\varphi_B([f]) = \{f^*(\gamma^k)\}$ for each $[f] \in [B, G_k(R^\infty)]$. Then the following proposition holds:

Proposition 2.3

$$\varphi = \{\varphi_B \mid B \in PS\} : [\text{——}, G_k(R)] \to Vect_k(\text{——})$$

is an isomorphism as a natural transformation between cofunctors.

Proof. It is trivial that $[\text{—}, G_k(R^\infty)] : PS \to E_{ns}$. For a $[f] : B \to B'$ in PS, the following diagram commutes:

$$\begin{array}{ccc} [B', G_k(R^\infty)] & \xrightarrow{\ \varphi_{B'}\ } & Vect_k(B') \\ {\scriptstyle [[f], G_k(R^\infty)]}\downarrow & & \downarrow{\scriptstyle Vect_k([f])} \\ [B, G_k(R^\infty)] & \xrightarrow{\ \varphi_B\ } & Vect_k(B) \end{array}$$

Thus, φ is a natural transformation, *i, e.*, for $[g] \in [B', G_k(R^\infty)]$,

$$Vect_k([f]) \circ \varphi_{B'}([g]) = Vect_k([f])(\{g^*(\gamma^k)\}) = \{f^*g^*(\gamma^k)\},$$

$$\varphi_B \circ [[f], G_k(R^\infty)]([g]) = \varphi_B([gf]) = \{f^* g^*(\gamma^k)\}.$$

Furthermore, it is clear that φ is an isomorphism. ∎

The following facts are well known:

(i) For each positive integers k and n, $G_k(R^n)$ is compact and hausdorff ([7], [9]).

(ii) (Morita's Theorem) If a regular topological space is the countable union of compact subsets, then it is paracompact and the direct limit of a sequence $K_1 \subset K_2 \subset \cdots$ of compact spaces is paracompact. Therefore, the direct limit $G_k(R^\infty)$ of a sequence $G_k(R^n) \subset G_k(R^{n+1}) \subset G_k(R^{n+2}) \subset \cdots$ of compact spaces is a paracompact space.

Definition 2.4 A real vector bundle $\xi_o = (E, \pi, B_o)$, where B_o is paracompact, is called a ***universal bundle*** if, for each paracompact space B and a real vector bundle $\eta^k = (E(\eta^k), \pi, B(\eta^k))$, there exists a continuous map $f: B \to B_o$ such that $\eta^k = f^*(\xi_o)$ for dimension k.

Example 2.5 (i) By Proposition 2.3, $\gamma^k = (E(\gamma^k), \pi, G_k(R^\infty))$ is a universal bundle for dimension k.

(ii) We can generalize the above example. Let $\mathscr{H}$ be a separable real Hilbert space. Since $\mathscr{H}$ has an orthonormal base, there exist n-dimensional subspaces in $\mathscr{H}$ for some nonnegative integer n. Define BGL_n to be the set of n-dimensional subspaces of $\mathscr{H}$, and topologize this set in some reasonable way, *i. e.*, define a metric on BGL_n by taking the distance between two n-dimensional subspaces of $\mathscr{H}$ to be the angle between these subspaces. Then for all $B \in PS$,

$$[B, BGL_n] \longleftrightarrow Vect_n(B)$$

([1]). Since BGL_n is a paracompact space,

$$\{(V, x) \in BGL_n \times \mathscr{H} \mid x \in V\} \longrightarrow BGL_n$$

is a universal bundle for dimension n. The space BGL_n is called a *classifying space* in the category PS, and for each $B \in PS$, a continuous map $f: B \longrightarrow BGL_n$ is called a *classifying map*.

In this paper, we need the concept of connections, and thus we shall describe some elementary properties about connections.

Definition 2.6 Let M be a differentiable manifold with dimension n, and let $T(M)$

$\to M$ be the tangent bundle over M. Then the fiber (or the tangent space) $T_x(M)$ at x is an n-dimensional real vector space. We put that

(i) $\mathcal{F}(M)$ = the set of all C^∞-class functions : $M \to R$,

(ii) $\mathcal{X}(M)$ = the set of all C^∞-class vector fields of M.

If $\pi : E \to M$ is a real vector bundle over M, we put

(iii) $\Gamma(E)$ = the set of all C^∞-class sections of M.

Then $\mathcal{X}(M) = \Gamma(T(M))$, $\mathcal{X}(M)$ and $\Gamma(E)$ are $\mathcal{F}(M)$-modules.

Definition 2.7 Let $\pi : E \to M$ be a real vector bundle. A *connection* $\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$ is an R-linear map satisfying

(i) $\nabla(X, fs) = \nabla_X(fs) = X(f)(s) + f\nabla_X(s)$,

(ii) $\nabla_{fX}(s) = f\nabla_X(s)$

for all $f \in \mathcal{F}(M)$, $X \in \mathcal{X}(M)$ and $s \in \Gamma(E)$. The *curvature* K of a connection ∇ is a map

$$K : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow Hom_R(\Gamma(E), \Gamma(E))$$

defined by

$$K(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for all $(X, Y) \in \mathcal{X}(M) \times \mathcal{X}(M)$.

Then, as is well known, the following hold ([1]):

(i) There exists at least one connection on a real vector bundle $E \to M$.

(ii) For $f, g, h \in \mathcal{F}(M)$, $X, Y \in \mathcal{X}(M)$ and $s \in \Gamma(E)$,
$K(fX, gY)(hs) = fghK(X, Y)(s)$.

(iii) For $X, Y \in \mathcal{X}(M)$, $K(X, Y) = -K(Y, X)$.

Consider a tangent bundle $T(M^n) \to M^n$ and a chart (U_j, φ_j).
Then for a local coordinate $(x^1, \cdots, x^n)$ $(x \in U_j,\ \varphi_j(x) = (x^1, \cdots, x^n))$, $T_x(M^n)$ has a base $\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right\}$. Let the dual base of $\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right\}$ be $\{dx^1, \cdots, dx^n\}$, which is a base of the n-dimensional real vector space $T_x(M)^*$. Let $\Lambda_n^*(U_j)$ be the associative algebra over R generated by $1, dx^1, \cdots, dx^n$ satisfying the following conditions:

(i) 1 is the multiplicative identity,

(ii) $dx^i \cdot dx^j = -dx^j \cdot dx^i$ for all $1 \leqq i, j \leqq n$.

We put

$$\Lambda_n^*(U_j) = \bigoplus_{k=0}^{n} \Lambda_n^k(U_j),$$

$$A^k(U_j) = \{w : U_j \to \Lambda_n^k(U_j) \mid w \text{ is smooth}\},$$

then $A^*(U_j)=\bigoplus_{k=0}^{n} A^k(U_j)$ is called the ***graded algebra*** of differential forms on U_j. Each element w of $A^k(U_j)$ is called a *k-form* on U_j, and it is written by

$$w=\sum_{1\leq i_1\leq\cdots\leq i_k\leq n} f_{i_1\cdots i_k}\, dx^{i_1}\cdots dx^{i_k},$$

where $f_{i_1\cdots i_k}\in\mathscr{F}(M^n)$.

Let $E\to M$ be a q-dimensional real vector bundle over a differentiable manifold M and let $(U_\alpha, \varphi_\alpha)$ be a chart of M. Then $E|U_\alpha$ has a smooth frame $\{s_\alpha^1, \cdots, s_\alpha^q\}$. Since $K(X, Y): \Gamma(E)\to\Gamma(E)$ is an R-linear map, $K(X, Y)$ is completely determined by $K(X, Y)(s_\alpha^1), \cdots,$ and $K(X, Y)(s_\alpha^q)$. Since $K(X, Y)(s_\alpha^i)$ $(1\leqq i\leqq q)$ is an element of the vector space generated by $\{s_\alpha^1, \cdots, s_\alpha^q\}$,

$$K(X, Y)(s_\alpha^i)=\sum_{j=1}^{q} K_{ji}^\alpha(X, Y)\, s_\alpha^j,$$

where $K_{ji}^\alpha(X, Y)\in R$. This implies that $K_{ji}^\alpha(X, Y)\in A^2(U_\alpha)$. Put $K_{ji}^\alpha(X, Y)=K_{ji}^\alpha$, then $K(X, Y)|U_\alpha=K^\alpha=(K_{ji}^\alpha)$ is called the *curvature matrix* of K (or ∇). In this case, we easily see that $K_{ji}^\alpha(X, Y)=-K_{ji}^\alpha(Y, X)$.

Proposition 2.8 For two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), $U_\alpha\cap U_\beta\neq\phi$, $K^\alpha=g_{\alpha\beta}K^\beta g_{\alpha\beta}^{-1}$, where $g_{\alpha\beta}=\varphi_\alpha^{-1}\circ\varphi_\beta: U_\alpha\cap U_\beta\to GL(n, R)$.

Proof If $\{s_\alpha^1, \cdots, s_\alpha^q\}$ is a smooth frame of $E|U_\alpha$, then $\{g_{\beta\alpha}(x)s_\alpha^1, \cdots, g_{\beta\alpha}(x)s_\alpha^q\}$ is a smooth frame of $E|U_\beta$, where $x\in U_\alpha\cap U_\beta$. Then

$$g_{\beta\alpha}^{-1}(x)K^\beta(X, Y)(g_{\beta\alpha}(x)s_\alpha^i)=K^\alpha(X, Y)(s_\alpha^i),$$

and therefore

$$g_{\beta\alpha}^{-1}(K^\beta)g_{\beta\alpha}=(K^\alpha).$$

By Proposition 2.1, $g_{\beta\alpha}^{-1}=g_{\alpha\beta}$. Thus we have

$$g_{\alpha\beta}(K^\beta)g_{\alpha\beta}^{-1}=g_{\beta\alpha}^{-1}K^\beta g_{\beta\alpha}=K^\alpha. \qquad ∎$$

Similarly, for $X\in\mathfrak{X}(M)$, $\nabla_X: \Gamma(E)\to\Gamma(E)$ is an R-linear map, and thus

$$\nabla_X(s_\alpha^i)=\sum_{j=1}^{q}\theta_{ij}^\alpha(X)\, s_\alpha^j,$$

where $\theta_{ij}^\alpha(X)\in A^1(U_\alpha)$. If we put $\theta^\alpha=(\theta_{ij}^\alpha(X))=(\theta_{ij}^\alpha)$, then $d\theta^\alpha-\theta^\alpha\cdot\theta^\alpha=K^\alpha$ ([8], vol. 1).

§ 3. Topological Categories

Definition 3.1 Let $\mathscr{C}$ be a category with the class $Obj(\mathscr{C})$ of objects and the class $Morph(\mathscr{C})$ of morphisms. A category $\mathscr{C}=Obj(\mathscr{C})\cup Morph(\mathscr{C})$ is called a *topological category* if it satisfies the following axioms:

(i) $Obj(\mathscr{C})$ and $Morph(\mathscr{C})$ are topological spaces.

(ii) The function $Morph(\mathscr{C}) \longrightarrow Obj(\mathscr{C}) \times Obj(\mathscr{C})$ defined by $f \rightsquigarrow (X, Y)$ for all $f \in Hom(X, Y)$ is continuous.

(iii) The composition $Morph(\mathscr{C}) * Morph(\mathscr{C}) \rightarrow Morph(\mathscr{C})$ is continuous, where $Morph(\mathscr{C}) * Morph(\mathscr{C}) = \{(f, g) \in Morph(\mathscr{C}) \times Morph(\mathscr{C}) \mid f \circ g \text{ is defined}\}$.

(iv) The function $Obj(\mathscr{C}) \rightarrow Morph(\mathscr{C})$ defined by $X \rightsquigarrow 1_X$ is continuous.

Therefore, the topological category is small; conversely, any small category $\mathscr{C}$ can be regarded as a topological category with the discrete topologies on $Obj(\mathscr{C})$ and $Morph(\mathscr{C})$.

Example 3.2 Let G be a topological group, and let $\mathscr{C}$ be a category such that $Obj(\mathscr{C}) = \{G\}$ and $Morph(\mathscr{C}) = G$ i.e., $\mathscr{C} = G \cup G$.

Then $\mathscr{C}$ is a topological category.

Proof $Obj(\mathscr{C}) = \{G\}$ has the discrete topology, while $Morph(\mathscr{C}) = G$ has the same topology as G. Since G is a topological group, the product of elements in G is continuous, and thus (ii) of Definition 3.1 is satisfied. The conditions (i) and (iii) are clear, since $Obj(\mathscr{C})$ has the discrete topology.

Example 3.3 Let X be a topological space, and let $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ be an open covering of X. We shall define the category $X_{\mathscr{U}}$ as follows:

For $\Sigma = \{\alpha, \beta, \cdots, \delta\} \subset A$, we put

$$U_\Sigma = U_\alpha \cap U_\beta \cap \cdots \cap U_\delta = \bigcap_{\alpha \in \Sigma} U_\alpha,$$

and

$$Obj(X_{\mathscr{U}}) = \{(\Sigma, x) \mid \Sigma \subset A,\ x \in U_\Sigma \neq \phi\}.$$

The set $Hom((\Sigma, x), (\Sigma', x'))$ of all morphisms from (Σ, x) to (Σ', x') is empty if either $x \neq x'$ or $U_\Sigma \not\subset U_{\Sigma'}$. Otherwise, it is the single inclusion $(U_\Sigma, x) \hookrightarrow (U_{\Sigma'}, x)$. In this case,

$$Morph(X_{\mathscr{U}}) = \{Hom((\Sigma, x), (\Sigma', x)) \mid (\Sigma, x), (\Sigma', x) \in Obj(X_{\mathscr{U}})\}.$$

The topology of $Obj(X_{\mathscr{U}})$ is obtained by taking, as basic neighborhoods, sets of the form

$$\{(\Sigma, x) \mid \Sigma \text{ fixed, } x \in W \subset U_\Sigma,\ W \text{ open in } U_\Sigma\}.$$

Similarly, the topology of $Morph(X_{\mathscr{U}})$ is defined by taking, as basic neighborhoods, sets of the form

$$\{(\Sigma, x) \hookrightarrow (\Sigma', x) \mid \Sigma \text{ and } \Sigma' \text{ fixed, } x \in W \subset U_\Sigma,\ W \text{ open in } U_\Sigma\}.$$

Naturally, in this case we have $U_\Sigma \subset U_{\Sigma'}$. And the category X_ν is a topological category.

Proof Let $f: Morph(X_\nu) \to Obj(X_\nu) \times Obj(X_\nu)$ be defined by

$$(\Sigma,\ x) \hookrightarrow (\Sigma',\ x) \rightsquigarrow ((\Sigma,\ x),\ (\Sigma',\ x)),$$

and let basic open sets of $(\Sigma,\ x)$ and $(\Sigma',\ x)$ be

$$V_{(\Sigma,x)} = \{(\Sigma,\ x) \mid x \in W \subset U_\Sigma,\ W \text{ open in } U_\Sigma\},$$

and

$$V_{(\Sigma',x)} = \{(\Sigma',\ x) \mid x \in W' \subset U_{\Sigma'},\ W' \text{ open in } U_{\Sigma'}\},$$

respectively. Then

$$f^{-1}((V_{(\Sigma,x)},\ V_{(\Sigma',x)})) = \{(\Sigma,\ x) \hookrightarrow (\Sigma',\ x) \mid x \in W \subset U_\Sigma,\ W \text{ open in } U_\Sigma\},$$

which is an open set of $(\Sigma,\ x) \hookrightarrow (\Sigma',\ x)$; and thus f is continuous.
For $i: (\Sigma,\ x) \hookrightarrow (\Sigma',\ x) \in Morph(X_\nu)$, $i^{-1}(V_{(\Sigma',x)}) = V_{(\Sigma,x)}$, and thus i is continuous.

In $Morph(X_u)$, if $(\Sigma,\ x) \xrightarrow{i_1} (\Sigma',\ x) \xrightarrow{i_2} (\Sigma',\ x)$, then $i_2 \circ i_1$ is continuous. Moreover, it is clear that

$$(\Sigma,\ x) \rightsquigarrow (\Sigma,\ x) \overset{1_{(\Sigma,x)}}{\hookrightarrow} (\Sigma,\ x)$$

is continuous. Therefore the category X_ν is a topological category. ∎

In order to make one more example (Example 3. 12) of a topological category, we need to go through further study.

Let R^q be the q-dimensional Euclidean space. A local diffeomorphism $f: R^q \to R^q$ is a continuous map such that there exists an open subset $U \subset R^q$ such that $f|U$ is a diffeomorphism. We put

$$\mathcal{D}(R^q) = \{\text{local diffeomorphisms}: R^q \to R^q\},$$

and for an open set $U \subset R^q$

$$\mathcal{D}(U) = \{f \in \mathcal{D}(R^q) \mid f|U \text{ is a diffeomorphism}\}.$$

For all $x \in R^q$, the *stalk* $\mathcal{D}_x$ at x is the set $\varinjlim_{x \in U} \mathcal{D}(U)$, where U is an open subset of R^q. Let γ^x be the germ of $\gamma \in \mathcal{D}(U)$ $(x \in U)$, and let $\mathcal{D} = \bigcup_{x \in R^q} \mathcal{D}_x$. Then $\mathcal{D}$ is a topological space with an open base

$$\{\{\gamma^x \mid \gamma \in \mathcal{D}(U),\ U \text{ open in } R^q\} \mid \text{for all open sets in } R^q\}.$$

The sheaf $\mathcal{D} \to R^q$ $(\mathcal{D}_x \rightsquigarrow x)$ (or $\mathcal{D}$) is called the *sheaf of germs of local diffeomorphisms* : $R^q \longrightarrow R^q$. For $x,\ y \in R^q$, we shall use the notation $\mathcal{D}_{(x,y)} = \{\gamma^x \in \mathcal{D}_x \mid \gamma^x(x) = y\}$.

Definition 3.4 A *Γ_q-cocycle* on a topological space X is defined by the following data: For an open cover $\{U_\alpha\}_{\alpha\in A}$ of X,

(i) there exists a continuous map $f_\alpha: U_\alpha \to R^q$ for all $\alpha \in A$.

We put $f_\alpha^x = \varinjlim_{x\in V\subset U_\alpha} f_\alpha|V$, where V is open in R^q.

(ii) For each $x\in U_\alpha\cap U_\beta$, a germ $\gamma_{\alpha\beta}^x \in \mathcal{J}_{(f_\beta x, f_\alpha x)}$ satisfies the following conditions (where α, β, $\delta\in A$):

(a) $U_\alpha\cap U_\beta \to \mathcal{J}$ $(x \mapsto \gamma_{\alpha\beta}^x)$ is continuous,

(b) $f_\alpha^x = \gamma_{\alpha\beta}^x \circ f_\beta^x$, and

(c) $\gamma_{\alpha\sigma}^x = \gamma_{\alpha\beta}^x \circ \gamma_{\beta\sigma}^x$ for each $x\in U_\alpha\cap U_\beta\cap U_\sigma$.

Furthermore, two Γ_q-cocycles $c = \{U_\alpha, f_\alpha, \gamma_{\alpha\beta}^x\}_{\alpha,\beta\in A}$ and $c' = \{U_\lambda, f_\lambda, \gamma_{\lambda\mu}^x\}_{\lambda,\mu\in B}$ are said to be ***equivalent*** if there exists a Γ_q-cocycle c'' corresponding to the cover $\{U_\varsigma\}_{\varsigma\in P}$, where P is the disjoint union of A and B, such that $c''|\{U_\alpha\}_{\alpha\in A} = c$ and $c''|\{U_\lambda\}_{\lambda\in B} = c'$.

Proposition 3.5 The above relation is an equivalence relation.

Proof It suffices to prove the transitive law. For three Γ_q-cocycles $a = \{U_\alpha, f_\alpha, \gamma_{\alpha\alpha'}^x\}_{\alpha,\alpha'\in A}$, $b = \{U_\beta, f_\beta, \gamma_{\beta\beta'}^x\}_{\beta,\beta'\in B}$ and $c = \{U_\sigma, f_\sigma, \gamma_{\sigma\sigma'}^x\}_{\sigma,\sigma'\in P}$, we shall prove that $a\sim b$ and $b\sim c$ implies $a\sim c$, where $\sim$ is the relation defined in the above. Since $a\sim b$ there exists a Γ_q-cocycle $a\cup b = \{U_\alpha, f_\alpha, \gamma_{\alpha\beta}^x\}_{\alpha,\beta\in A\cup B}$ such that

$$(a\cup b)\,|\,\{U_\alpha, f_\alpha, \gamma_{\alpha\alpha'}^x\}_{\alpha,\alpha'\in A} = a \text{ and}$$
$$(a\cup b)\,|\,\{U_\beta, f_\beta, \gamma_{\beta\beta'}^x\}_{\beta,\beta'\in B} = b.$$

Since $b\sim c$, there exists a Γ_q-cocycle $b\cup c = \{U_\beta, f_\beta, \gamma_{\beta\sigma}^x\}_{\beta,\sigma\in B\cup P}$ such that

$$(b\cup c)\,|\,\{U_\beta, f_\beta, \gamma_{\beta\beta'}^x\}_{\beta,\beta'\in B} = b \text{ and}$$
$$(b\cup c)\,|\,\{U_\sigma, f_\sigma, \gamma_{\sigma\sigma'}^x\}_{\sigma,\sigma'\in P} = c.$$

In this case, we define $a\cup c = \{U_\alpha, f_\alpha, \gamma_{\alpha\sigma}^x\}_{\alpha,\sigma\in A\cup P}$ such that for any $x\in U_\alpha\cap U_\sigma$ $(U_\alpha\in a, U_\sigma\in c)$,

$$\gamma_{\alpha\sigma}^x = \gamma_{\alpha\lambda}^x \circ \gamma_{\lambda\sigma}^x,$$

where $\gamma_{\alpha\lambda}^x \in a\cup b$, $\gamma_{\lambda\sigma}^x \in b\cup c$, $x\in U_\alpha\cap U_\lambda\cap U_\sigma$ and $U_\lambda\in b$. For the other open set $U_\beta\in b$ with $x\in U_\alpha\cap U_\beta\cap U_\sigma$, it is easy to prove that

$$\gamma_{\alpha\beta}^x \circ \gamma_{\beta\sigma}^x = \gamma_{\alpha\lambda}^x \circ \gamma_{\lambda\sigma}^x,$$

and thus $\gamma_{\alpha\sigma}^x$ is well-defined. Clearly,

$$(a\cup c)\,|\,\{U_\alpha, f_\alpha, \gamma_{\alpha\alpha'}^x\}_{\alpha,\alpha'\in A} = a \text{ and}$$
$$(a\cup c)\,|\,\{U_\sigma, f_\sigma, \gamma_{\sigma\sigma'}^x\}_{\sigma,\sigma'\in P} = c. \quad ∎$$

We put $H(X;\ \Gamma_q)$ = the set of all Γ_q-cocycles on X, and

$$H^1(X;\ \Gamma_q)=H(X;\Gamma_q)/\sim$$

Definition 3.6 A *foliation* E of a differentiable (smooth) manifold M is a smooth subbundle of $T(M)$ satisfying the following conditions:

(i) There exists an open covering $\{U_\alpha\}_{\alpha\in A}$ of M,

(ii) there exists submersions $\{f_\alpha : U_\alpha \to R^q\}_{\alpha\in A}$ such that

(a) $E|U_\alpha = Ker\ df_\alpha$ for all $\alpha\in A$,

(b) $Ker\ df_\alpha|U_\alpha\cap U_\beta = Ker\ df_\beta|U_\alpha\cap U_\beta$ for all $\alpha,\beta\in A$.

That is, in the commutative diagram

$$\begin{array}{ccc} T(U_\alpha) & \xrightarrow{\ df_\alpha\ } & T(R^q) \\ \downarrow & & \downarrow \\ U_\alpha & \xrightarrow{\ f_\alpha\ } & R^q \end{array},$$

we note that $Ker\ df_\alpha = E|U_\alpha$. (Sometimes, E is said to be q-dimensional).

Proposition 3.7 Let E be a foliation with dimension q. Then E defines a unique element in $H^1(M;\Gamma_q)$.

Proof Let $\{(U_\alpha,\ \varphi_\alpha)\}_{\alpha\in A}$ be a local coordinate system. For each $x\in U_\alpha\cap U_\beta(\alpha,\beta\in A)$, we have homeomorphisms such that the diagram commutes:

$$\begin{array}{ccccc} & \overset{h_\alpha|x\times R^n}{\nearrow \cong} & T(M)_x & \xrightarrow[\cong]{\ d\varphi_{\alpha,x}\ } & R^n \\ x\times R^n & \copyright & \cong\downarrow h_{\alpha\beta}|\pi^{-1}(x) & \copyright & \cong\downarrow \bar{\gamma}_{\beta\alpha} \\ & \underset{h_\beta|x\times R^n}{\searrow \cong} & T(M)_x & \xrightarrow[\cong]{\ d\varphi_{\beta,x}\ } & R^n \end{array},$$

where $(T(M),\ \pi,\ M)$ is the tangent bundle over M (see Proposition 2.1). Note that every homeomorphism in the above diagram is smooth. Since $E_x\cong R^{n-q}$, we have the commutative diagram:

$$\begin{array}{ccccc} T(M)_x & \xrightarrow{\ p_r\ } & T(M)_x/E_x & \xrightarrow[\cong]{\ \bar{f}_\alpha\ } & R^q \\ h_{\alpha\beta}|\pi^{-1}(x)\downarrow\cong & \copyright & \cong\downarrow & \copyright & \downarrow\gamma^x_{\beta\alpha} \\ T(M)_x & \xrightarrow{\ p_r\ } & T(M)_x/E_x & \xrightarrow[\cong]{\ \bar{f}_\beta\ } & R^q \end{array}$$

where p_r denotes the canonical projection. we define $f_\alpha : U_\alpha \to R^q$ and $f_\beta : U_\beta \to R^q$ by compositions

$$f_\alpha(x) = \bar{f}_\alpha \circ p_r \circ (h_\alpha | x \times R^n)(x,\ \varphi_\alpha(x)) \quad (x \in U_\alpha),$$

$$f_\beta(y) = \bar{f}_\beta \circ p_r \circ (h_\beta | y \times R^n)(y,\ \varphi_\beta(y)) \quad (y \in U_\beta).$$

Then it is clear that for each $x \in U_\alpha \cap U_\beta$,

$$\gamma^x_{\alpha\beta} \circ f^x_\beta = f^x_\alpha \quad \text{i.e.} \quad \gamma^x_{\alpha\beta} = \lambda_{(f_\beta(x),\ f_\alpha(x))}.$$

By our constructions of f_α, f_β and $\gamma_{\alpha\beta}$, if we take another local coordinate system, $\gamma^x_{\alpha\beta}$ is coincided. Therefore, a given foliation defines a unique element of $H^1(M; \Gamma_q)$.

∎

In general, for two Γ_q-cocycles $c = \{U_\alpha,\ f_\alpha,\ \gamma^x_{\alpha\beta}\}_{\alpha,\beta \in A}$ and $c' = \{U_\alpha,\ g_\alpha,\ \bar{\gamma}^x_{\alpha\beta}\}_{\alpha,\beta \in A}$, if $g_\alpha \simeq f_\alpha$ (homotopic) for all $\alpha \in A$, it is not always true that $c = c'$. Hence we need the following definition.

Definition 3.8 For $c,\ c' \in H^1(X; \Gamma_q)$, we say that c and c' are ***homotopic*** (written $c \simeq c'$) if there exists $c'' \in H^1(X \times I; \Gamma_q)$ such that $c = i_0^*(c'')$ and $c' = i_1^*(c'')$, where I is the closed unit interval and $i_0,\ i_1 : X \to X \times I$ are defined by $i_0(x) = (x,\ 0)$ and $i_1(x) = (x,\ 1)$ for all $x \in X$. We may put

$$\Gamma_q(X) = H^1(X; \Gamma_q) / \simeq,$$

since we can easily prove that "$\simeq$" is an equivalence relation.

By [5] and [6], we can construct a space $B\Gamma_q$ such that

$$\Gamma_q(-) \cong [\,-\!-,\ B\Gamma_q].$$

Therefore, for every CW-complex X, we have

$$\Gamma_q(X) \cong [X,\ B\Gamma_q]$$

as sets. That is, $B\Gamma_q$ is a classifying space in the category of CW-complexes for $\Gamma_q(X)$ ([6]).

Let M be a smooth manifold. An element $c = \{U_\alpha,\ f_\alpha,\ \gamma^x_{\alpha\beta}\}_{\alpha,\beta \in A}$ is said to be ***smooth*** if f_α and $\gamma_{\alpha\beta}$ are smooth for all $\alpha,\ \beta \in A$. By Proposition 3.7, there is a bijection between the set of all smooth Γ_q-cocycles in $H^1(M; \Gamma_q)$ and the set of all foliations of M.

Definition 3.10 Let $H^k_{DR}(M)$ be the k-dimensional ***de Rham cohomology group*** of M, and let φ be an ***invariant polynomial*** of degree r ([1], [7]). Then $\varphi(E) = [\varphi(K)] \in H^{2r}_{DR}(M)$ is called the ***Pontryagin class*** ([10]) corresponding to φ, where $\pi : E \to M$ is a real vector bundle and K is the curvature of a connection on E (see

§ 2.). The *i-th* Pontryagin class $P_i(E)$ is in $H^{4i}_{DR}(M)$, the graded subring $Pont^*(E)$ = the subring of $H^*_{DR}(M)$ generated by $\{1, P_1(E), \cdots, P_{[\frac{n}{2}]}(E)\}$, where $\pi: E \to M$ is n-dimensional.

Proposition 3.11 If N is the normal bundle corresponding to a smooth Γ_q-cocycle c on M, then $Pont^k(N)=0$ for $k>2q$.

Proof If φ is an invariant polynomial, it is well known that

$$d(\varphi(K))=d(\varphi(d\theta^\alpha-\theta^\alpha\cdot\theta^\alpha))=0$$

(see § 2.) and $[\varphi(K)]\in H^{2r}_{DR}(M)$ is independent of the choice of the connection. In particular, $Pont^j(E)=0$ if j is not divisible by 4 ([1]). Now, since $N\simeq T(M)/E$ (i.e., $T(M)=E\oplus N$) for a foliation E of $T(M)$, there exists an open covering $\{U_\alpha\}_{\alpha\in A}$ of M and the Γ_q-cocycle $\{U_\alpha, f_\alpha, \gamma^x_{\alpha\beta}\}_{\alpha,\beta\in A}$ corresponding to E. Let ∇^q be the connection obtained by pulling back the standard connection on R^q by f_α, where the standard connection on R^q is defined by

$$\nabla_X(Y)=XY \quad \text{for} \quad X, Y\in \mathfrak{X}(R^q)$$

i.e., if

$$X_x=\sum_{\nu=1}^{q} X^\nu(x)\frac{\partial}{\partial x^\nu} \quad \text{and} \quad Y_x=\sum_{\mu=1}^{q} Y^\mu(x)\frac{\partial}{\partial x^\mu}$$

for all $x=(x^1, \cdots, x^q)\in R^q$, then

$$X_xY_x=\sum_{\mu=1}^{q}\left(\sum_{\nu=1}^{q} X^\nu(x)\frac{\partial Y^\mu(x)}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu}.$$

Also, we note that, for $x\in U_\alpha$ and a vector field $X_x=\sum_{\nu=q+1}^{n} X^\nu(x)\frac{\partial}{\partial x^\nu}$ $(x=(x^1, \cdots, x^q, x^{q+1}, \cdots, x^n))$ of N at x, the vector field of R^q at $f_\alpha(x)=(f^1_\alpha(x^1, \cdots, x^n), \cdots, f^q_\alpha(x^1, \cdots, x^n))=(y^1, \cdots, y^q)$ is

$$\sum_{i=1}^{q}\left(\sum_{\nu=1}^{n} X^\nu(x)\frac{\partial y^i}{\partial x^\nu}\right)\frac{\partial}{\partial y^i}.$$

Since M is paracompact, there exists a partition of unity $\{\lambda_\alpha\}_{\alpha\in A}$ of $\{U_\alpha\}_{\alpha\in A}$, and we have the connection $\nabla=\sum_{\alpha\in A}\lambda_\alpha\nabla^\alpha$ of N. For each $U_\alpha(\alpha\in A)$ and the curvature matrix K^α of ∇, K^α_U is in the ideal of $A^*(U_\alpha)$ generated by those of 1-forms which are local pull-backs by f_α of 1-forms on R^q. The dimension of N is q, and thus $I_\alpha(\gamma)^{q+1}=0$. This means that $\varphi(K)=0$ for an invariant polynomial φ with degree greater than q. ∎

Example 3.12 We want to construct the topological category Γ_q as follows: we put $Obj(\Gamma_q)=R^q$ with the usual topology (i.e., each object of Γ_q is a point in R^q) and $Morph(\Gamma_q)=\{\mathcal{J}_{(x,y)} \mid (x, y)\in R^q\times R^q\}$. Therefore $Morph(\Gamma_q)=\mathcal{J}$ has the sheaf topology. Then the category Γ_q is a topological category.

Proof At first, we shall prove the continuity of $\mathcal{I} * \mathcal{I} \to \mathcal{I}((\gamma_1, \gamma_2) \mapsto \gamma_2 \circ \gamma_1)$. If $\gamma_1 \in \mathcal{I}_{(x,y)}$ and $\gamma_2 \in \mathcal{I}_{(y,z)}$, then there exist open sets U_1 containing x and U_2 containing y such that $\gamma_1 | U_1$ and $\gamma_2 | U_2$ are diffeomorphisms. We put

$$\{\gamma_1(x) \mid x \in U_1\} \cap U_2 = U \quad \text{and} \quad \gamma_1^{-1}(U) = V,$$

then $\{(\gamma_2 \circ \gamma_1)^z \mid z \in \gamma_2(U)\}$ is an open neighborhood of $\gamma_2 \circ \gamma_1$. Note that U, V and $\gamma_2(U)$ are open in R^q. Then

$$(\{\gamma_1^x \mid x \in V\},\ \{\gamma_2^y \mid y \in U\})$$

is an open set in $\mathcal{I} \times \mathcal{I}$ corresponding to the open set $\{(\gamma_2 \cdot \gamma_1)^z \mid z \in \gamma_2(U)\}$, where $(\gamma_2 \circ \gamma_1)^z = \gamma_2^y \circ \gamma_1^x$ such that $\gamma_1(x) = y$ and $\gamma_2(y) = z$. Thus the continuity is proved. Also,

$$Morph(\Gamma_q) \to Obj(\Gamma_q) \times Obj(\Gamma_q) \quad (\mathcal{I}_{(x,y)} \ni \gamma \mapsto (x, y))$$

is continuous, because there exists an open set U containing x such that $\gamma | U$ is a diffeomorphism and $(U, \gamma(U)) \mapsto \{\gamma^x \mid x \in U\}$, where $\gamma(U)$ is an open neighborhood of y.

Finally, we shall prove that $Obj(\Gamma_q) \to Morph(\Gamma_q)$ $(x \mapsto 1_x)$ is continuous. Since 1_x is the germ of $1_{R^q}: R^q \to R^q$ at x, $\{1_x^y \mid y \in U\} \cong U$ for an open set U containing x. Therefore, $Obj(\Gamma_q) \to Morph(\Gamma_q)$ $(x \mapsto 1_x)$ is continuous. ∎

§ 4. Pseudo-universal Spaces

Definition 4.1 For a given toplogical space X and an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of X, if the set

$$\{g_{\beta\alpha}: U_\alpha \cap U_\beta \to GL(q, R) \mid \alpha, \beta \in A,\ g_{\beta\alpha} \text{ is continuous}\}$$

satisfies the following conditions:

(i) $g_{\alpha\alpha}(x) =$ identity matrix for $x \in U_\alpha$, and

(ii) $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$,

then each $g_{\alpha\beta}$ is called a *GL_q-cocycle*. For given GL_q-cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$, if there exists a collection of continuous maps

$$\{\theta_\alpha: U_\alpha \to GL(q, R)\}$$

such that

$$g'_{\alpha\beta}(x) = \theta_\alpha(x) \circ g_{\alpha\beta}(x) \circ \theta_\beta(x)^{-1}$$

for all $x \in U_\alpha \cap U_\beta$ and all $(\alpha, \beta) \in A \times A$, then $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are said to be *equivalent*, written $\{g_{\alpha\beta}\} \sim \{g'_{\alpha\beta}\}$. Since the relation "$\sim$" is an equivalence relation, we put

$\{\text{all } GL_q\text{-cocycles } \{g_{\alpha\beta}\}\}/\sim = H^1(X_{\mathscr{U}}; GL_q)$.

The following proposition is well known ([7], [13]).

Proposition 4.2 Let ξ^n be a real vector bundle with a local coordinate system $\{(U_c, \varphi_\alpha)\}_{\alpha\in A}$. Then there exists a unique set $\{g_{\beta\alpha}: U_\alpha \cap U_\beta \to GL(n, R)\}_{\alpha,\beta\in A}$ of GL_q-cocycles. The converse is also true. Let η^n be a real vector bundle with the set $\{g'_{\beta\alpha}: U_\alpha \cap U_\beta \to GL(n, R)\}_{\alpha,\beta\in A}$ of GL_q-cocycles and the same base as ξ^n. If $\xi^n \cong \eta^n$, then there exists a set

$$\{\theta_\alpha: U_\alpha \to GL(n, R) \mid \theta_\alpha \text{ is continuous, } \alpha\in A\}$$

such that $g'_{\beta\alpha}(x) = \theta_\beta(x)\circ g_{\beta\alpha}(x)\circ\theta_\alpha(x)^{-1}$ for all $x\in U_\alpha\cap U_\beta$ and $\alpha, \beta\in A$.

Since $GL(q, R)$ is a topological group, we obtain the topological category GL_q by Example 3.2. Then, $Obj(GL_q)$ and $Morph(GL_q)$ are the same subset of $Morph(\Gamma_q)$ consisting of all diffeomorphisms R^q to itself.

Definition 4.3 Let $\mathscr{C}$ and $\mathscr{C}'$ be two topological categories. A functor

$$F: \mathscr{C} = Obj(\mathscr{C}) \cup Morph(\mathscr{C}) \longrightarrow \mathscr{C}' = Obj(\mathscr{C}') \cup Morph(\mathscr{C}')$$

is *continuous* if $F|Obj(\mathscr{C})$ and $F|Morph(\mathscr{C})$ are both continuous.

Example 4.4 We give a continuous functor $F: X_{\mathscr{U}} \to GL_q$ as follows: The topological category $X_{\mathscr{U}}$ is the same as Example 3.3, where $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ is a fixed open covering of X. Define F by taking

$$F((U_\Sigma, x) \to (U_{\Sigma'}, x)) = g_{\Sigma'\Sigma}(x),$$

where $x\in U_\Sigma \subset U_{\Sigma'}$ and $g_{\Sigma'\Sigma}(x) = g_{\beta\alpha}(x)$ is a GL_q-cocycle, α is the maximal element in Σ and β is the maximal element in Σ' in a linear ordering of A. Let us put

$$g_{\Sigma'\Sigma}(x) = \begin{pmatrix} g_{11}(x) & \cdots & g_{1q}(x) \\ \vdots & & \vdots \\ g_{q1}(x) & \cdots & g_{qq}(x) \end{pmatrix},$$

then there exists an open neighborhood V of x such that $V\subset U_\Sigma \subset U_{\Sigma'}$ and for every $y\in V$, we have

$$\begin{pmatrix} g_{11}(y) & \cdots & g_{1q}(y) \\ \vdots & & \vdots \\ g_{q1}(y) & \cdots & g_{qq}(y) \end{pmatrix} = g_{\Sigma'\Sigma}(y) \in GL(q, R).$$

This implies that $F|Morph(X_{\mathscr{U}})$ is continuous, where $g_{ij}: V \to R$ is continuous for

$1 \leqq i,\ j \leqq q$. Next, we put

$$F((U_\Sigma,\ x)) = GL(q,\ R),$$

since $Obj(GL_q) = \{GL(q,\ R)\}$. Then

$$F \mid Obj(X_{\mathscr{U}}) : Obj(X_{\mathscr{U}}) \longrightarrow Obj(GL_q)$$

is continuous.

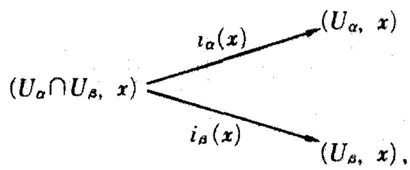
Furthermore, for $U_\Sigma \subset U_{\Sigma'} \subset U_{\Sigma''}$, the followings hold:

(i) $F(1_{(U_{\Sigma'},x)}) = g_{\Sigma\Sigma}(x) = 1_{GL(q,R)}$.

(ii) $F(((U_{\Sigma'},\ x) \to (U_{\Sigma''},\ x)) \circ ((U_\Sigma,\ x) \to (U_{\Sigma'},\ x)))$
$= F((U_\Sigma,\ x) \to (U_{\Sigma''},\ x)) = g_{\Sigma''\Sigma}(x) = g_{\Sigma''\Sigma'}(x) \circ g_{\Sigma'\Sigma}(x)$
$= F((U_{\Sigma''},\ x) \to (U_{\Sigma'},\ x)) \circ F((U_{\Sigma'},\ x) \to (U,\ x))$.

Therefore, $F: X_{\mathscr{U}} \longrightarrow GL_q$ is a continuous functor. ∎

In the category $X_{\mathscr{U}}$ with an open covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ of X, for $(\alpha,\ \beta) \in A \times A$ and $x \in U_\alpha \cap U_\beta$, there exist only two morphisms such that



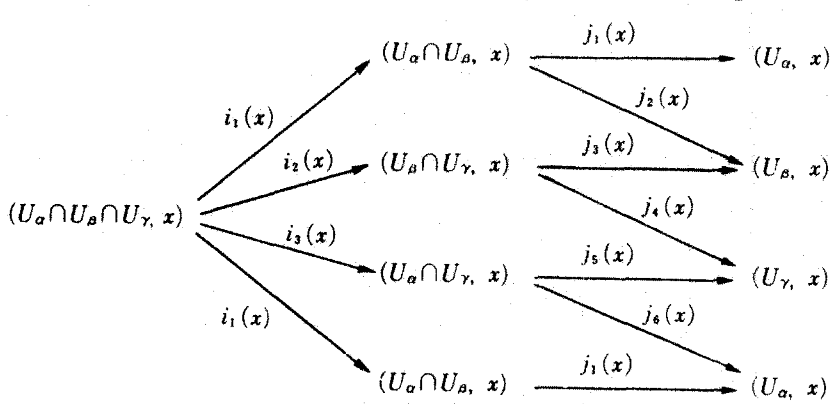
where $i_\alpha(x)$ and $i_\beta(x)$ are inclusion maps. Then, as the converse of Example 4.4, we have the following lemma.

Lemma 4.5 Let $F: X_{\mathscr{U}} \to GL_q$ be a continuous functor. Then

$$\{g_{\alpha\beta}(x) = F(i_\alpha(x)) \circ (F(i_\beta(x)))^{-1} \mid \alpha,\ \beta \in A \text{ and } x \in U_\alpha \cap U_\beta\}$$

is a set of GL_q-cocycles.

Proof Since $g_{\alpha\alpha}(x) = F(i_\alpha(x)) \circ (F(i_\alpha(x)))^{-1}$, $g_{\alpha\alpha}(x) = 1_{R^n}$ for all $\alpha \in A$ and $x \in U$. For $x \in U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$, there exists the following commutative diagram:



Then, we have

$$\begin{aligned} g_{\alpha\beta}(x) &= F(j_1(x))\circ(F(j_2(x)))^{-1} \\ &= F(j_1(x))\circ F(i_1(x))\circ(F(i_1(x)))^{-1}\circ(F(j_2(x)))^{-1} \\ &= F(j_1(x)\circ i_1(x))\circ(F(j_2(x)\circ i_1(x)))^{-1}, \\ g_{\beta\gamma}(x) &= F(j_3(x)\circ i_2(x))\circ(F(j_4(x)\circ i_2(x)))^{-1}, \end{aligned}$$

and thus

$$\begin{aligned} g_{\alpha\beta}(x)\circ g_{\beta\gamma}(x) &= F(j_1(x)\circ i_1(x))\circ(F(j_2(x)\circ i_1(x)))^{-1}\circ F(j_3(x)\circ i_2(x)) \\ &\quad \circ(F(j_4(x)\circ i_2(x)))^{-1} \\ &= F(j_1(x)\circ i_1(x))\circ(F(j_4(x)\circ i_2(x)))^{-1} \\ &= F(j_6(x)\circ i_3(x))\circ(F(j_5(x)\circ i_3(x)))^{-1} \\ &= F(j_6(x))\circ(F(j_5(x)))^{-1} = g_{\alpha\gamma}(x). \end{aligned}$$

Therefore, $\{g_{\alpha\beta}(x)=F(i_\alpha(x))\circ(F(i_\beta(x)))^{-1}\mid \alpha,\beta\in A,\ x\in U_\alpha\cap U_\beta\}$ satisfies the conditions (i) and (ii) in Definition 4.1. ∎

Definition 4.6 For two continuous functors F and L between topological categories $\mathscr{C}$ and $\mathscr{C}'$, a natural transformation $\theta: F\to L$ is said to be ***continuous*** if

$$\begin{array}{ccl} \theta: Obj(\mathscr{C}) & \to & Morph(\mathscr{C}') \\ \Psi & & \Psi \\ H & \rightsquigarrow & \theta(H): F(H)\to L(H) \end{array}$$

is continuous. Also, $F\cong L$ if there is a continuous natural transformation $\theta: F\to L$ which is an isomorphism.

Definition 4.7 Let $\mathscr{C}(TS)$ be a category of topological spaces and continuous maps. If a class $\mathscr{J}$ of sets satisfies the following conditions:

(i) There exists an injection $Vect_n(X)\to\mathscr{J}$ for each $X\in\mathscr{C}(TS)$ and for $n=0, 1, 2, \cdots$,

(ii) $\mathscr{J}\neq\{Vect_n(X)\}_{n=0,1,\cdots;\ X\in\mathscr{C}(TS)}$,

then $\mathscr{J}$ is called a ***pseudo-universal space*** of $\mathscr{C}(TS)$.

Let $\mathscr{C}(M)$ be the category of all differentiable manifolds and all manifold homomorphisms. Then, a q-dimensional real vector bundle over M has a local coordinate system $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$. We put $\mathscr{U}=\{U_\alpha\}_{\alpha\in A}$ which is an open covering of M. By $Iso(M_{\mathscr{U}}, GL_q)$, we mean the set of isomorphism classes of all continuous functors from $M_{\mathscr{U}}$ to GL_q. If we put $\overline{Iso}(\mathscr{C}(M))$ denoting

$$\{Iso(M_{\mathscr{U}}, GL_q)\mid q=0, 1, \cdots;\ \mathscr{U} \text{ an open covering of } M\in\mathscr{C}(M)\},$$

which may be a class of sets. We shall introduce the equivalence relation "$\sim$" in $\overline{Iso}(\mathscr{C}(M))$ by $Iso(M_{\mathscr{V}}, GL_q) \sim Iso(M_{\mathscr{V}'}, GL_p)$ if and only if $p=q$ and $Iso(M_{\mathscr{V}}, GL_q) \longleftrightarrow Iso(M_{\mathscr{V}'}, GL_p)$ as sets. And we define

$$Iso(\mathscr{C}(M)) = \overline{Iso}(\mathscr{C}(M))/\sim.$$

Theorem 4.8 $Iso(\mathscr{C}(M))$ is a pseudo-universal space of the category $\mathscr{C}(M)$.

To prove the theorem, we need a Lemma.

Lemma 4.9 For each $M \in Obj(\mathscr{C}(M))$ and an open covering $\mathscr{V} = \{U_\alpha\}_{\alpha \in A}$ of M, there exists an one-to-one correspondence between $Iso(M_{\mathscr{V}}, GL_q)$ and $H^1(\mathscr{V}, GL_q) = H^1(M_{\mathscr{V}}, GL_q)$.

Proof Let $F_1, F_2 \colon M_{\mathscr{V}} \to GL_q$ be continuous functors which are isomorphic. Then there exists a natural transformation $\theta \colon F_1 \to F_2$ which is an isomorphism. Then, for the diagram

$$\begin{array}{ccc} & & (U_\alpha, x) \\ & \nearrow^{i_\alpha(x)} & \\ (U_\alpha \cap U_\beta, x) & & \\ & \searrow_{i_\beta(x)} & \\ & & (U_\beta, x) \end{array}$$

in the topological category $M_{\mathscr{V}}$, we have the following commutative digram:

$$\begin{array}{ccc} GL(q, R) & \xrightarrow{\quad F_1(i_\alpha(x)) \quad} & GL(q, R) \\ {\scriptstyle \theta(U_\alpha \cap U_\beta, x)} \downarrow & & \downarrow {\scriptstyle \theta(U_\alpha, x)} \\ GL(q, R) & \xrightarrow{\quad F_2(i_\alpha(x)) \quad} & GL(q, R) \end{array}$$

This means that

$$\theta(U_\alpha, x) \circ F_1(i_\alpha(x)) = F_2(i_\alpha(x)) \circ \theta(U_\alpha \cap U_\beta, x).$$

Let $\{g^1_{\alpha\beta}\}$ and $\{g^2_{\alpha\beta}\}$ be the GL_q-cocycles which are defined by F_1 and F_2, respectively; in the same way as in Lemma 4.5. Then for each $x \in U_\alpha \cap U_\beta$,

$$\begin{aligned} g^1_{\alpha\beta}(x) &= F_1(i_\alpha(x)) \circ (F_1(i_\beta(x)))^{-1} \\ &= (\theta(U_\alpha, x))^{-1} \circ F_2(i_\alpha(x)) \circ \theta(U_\alpha \cap U_\beta, x) \circ (\theta(U_\alpha \cap U_\beta, x))^{-1} \\ &\quad \circ (F_2(i_\beta(x)))^{-1} \circ \theta(U_\beta, x) \end{aligned}$$

$$= (\theta(U_\alpha, x))^{-1} \circ F_2(i_\alpha(x)) \circ (F_2(i_\beta(x)))^{-1} \circ \theta(U_\beta, x)$$
$$= (\theta(U_\alpha, x))^{-1} \circ g^2_{\alpha\beta}(x) \circ \theta(U_\beta, x).$$

We put $(\theta(U_\alpha, x))^{-1} = \theta_\alpha(x)$, then we have

$$g^1_{\alpha\beta}(x) = \theta_\alpha(x) \circ g^2_{\alpha\beta}(x) \circ (\theta_\beta(x))^{-1}.$$

This implies that $Iso(M_{\mathscr{U}}, GL_q)$ is mapped into $H^1(\mathscr{U}, GL_q)$ in a well-defined way.

Example 4.4 shows that, for each $\{g_{\alpha\beta}\}$ of GL_q-cocycles there exists a continuous functor $F: M_{\mathscr{U}} \to GL_q$. Let F_1 and F_2 be obtained from equivalent GL_q-cocycles $\{g^1_{\alpha\beta}\}$ and $\{g^2_{\alpha\beta}\}$, respectively; where there exist continuous maps $\theta_\alpha: U_\alpha \longrightarrow GL(q, R)$ such that $g^1_{\alpha\beta} = \theta_\alpha \circ g^2_{\alpha\beta} \circ \theta_\beta^{-1}$. We have then the following commutative diagram:

$$\begin{array}{ccc} GL(q, R) & \xrightarrow{\;F_1((U_\Sigma, x) \to (U_{\Sigma'}, x)) = g^1_{\Sigma'\Sigma}(x)\;} & GL(q, R) \\ \theta(U_\Sigma, x) \Big\downarrow & & \Big\downarrow \theta(U_{\Sigma'}, x) \\ GL(q, R) & \xrightarrow{\;F_2((U_\Sigma, x) \to (U_{\Sigma'}, x)) = g^2_{\Sigma'\Sigma}(x)\;} & GL(q, R) \end{array}$$

If α is a maximal element in the finite set $\Sigma' \subset A$ and β a maximal element in the finite set $\Sigma \subset A$ for some linear order of A, then

$$g^1_{\Sigma'\Sigma}(x) = g^1_{\alpha\beta}(x), \qquad g^2_{\Sigma'\Sigma}(x) = g^2_{\alpha\beta}(x);$$
$$\theta(U_{\Sigma'}, x) = \theta_\alpha(x), \qquad \theta(U_\Sigma, x) = \theta_\beta(x).$$

(see Example 4.4). This implies that $\theta_\alpha: U_\alpha \to GL(q, R)$ is an isomorphic natural transformation between F_1 and F_2. Since θ_α is continuous, we have $F_1 \cong F_2$. Thus, there is a one-to-one correspondence between $Iso(M_{\mathscr{U}}, GL_q)$ and $H^1(\mathscr{U}, GL_q)$.

Proof of Theorem 4.8 For $M \in \mathscr{C}(M)$, we fix an open covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ of M. The set of isomorphism classes of q-dimensional real vector bundles over M with a local neighborhood system $\mathscr{U}$ is obtained by $Vect_{q,\mathscr{U}}(M)$. Then, by Proposition 4.2 and Lemma 4.9, there is a one-to-one correspondence between $Vect_{q,\mathscr{U}}(M)$ and $H^1(U; GL_q)$. This means that $Iso(\mathscr{C}(M))$ is a pseudo-universal space of the category $\mathscr{C}(M)$. ∎

Note that if we introduce an equivalence relation "$\sim$" in

$$GL = \{H^1(\mathscr{U}; GL_q) \mid q = 0, 1, \cdots, \mathscr{U}: \text{open cover of } M \in \mathscr{C}(M)\}$$

by $H^1(\mathscr{U}, GL_q) \sim H^1(\mathscr{U}', GL_p)$ if and only if $p = q$ and $H^1(\mathscr{U}, GL_q) \longleftrightarrow H^1(\mathscr{U}', GL_p)$ as sets, then $UGL = GL/\sim$ is also a pseudo-universal space of the category $\mathscr{C}(M)$.

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ABSTRACT

多樣體의 研究에서 벡터束의 槪念은 不可缺하며 벡터束의 研究에는 葉層構造의 研究가 重要하다.

本 論文은 圈 $\mathscr{C}(M)$의 擬似普遍空間에 관한 研究(定理 4.8)로서 R. Bott의 葉層構造에 관한 研究([1])에서 着想된 것이다.

第二, 三節은 第四節을 위한 準備로서, 第二節에서는 벡터束 및 接續에 관한 性質을 논하고, 第三節에서는 位相圈, 層, 葉層構造 및 Γ_q-cocycle 등에 관한 性質(命題 3.5, 3.7과 3.11)을 밝히고, 位相圈의 具體的인 例(例 3.2, 3.3과 3.12)를 들었다.

第四節에서는, GL_q-cocycle, 位相圈 GL_q, 集合 $I_{so}(M_\#, GL_q)$, $H^1(M_\#, GL_q)$, $I_{so}(\mathscr{C}(M))$ 및 擬似普遍空間을 定義하고, 主定理 4.8의 證明에 必要한 命題를 몇개 記述하였다(命題 4.2, 4.5와 補題 4.9).