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THE GENERAL WEIGHTED ORLICZ-SOBOLEV SPACES

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Let A be a Banach space and let $[X, \beta, \mu]$ be a measure space with σ-finite positive measure μ. If $0<w(x)<\infty$ is a measurable function, and $1\leqq p<\infty$, then we put

$$L_{p,w}(A)=L_{p,w}(A;X,\beta,\mu)$$
$$=\{f;\ f(x)\in A,\ \|f\|_{L_{p,w}(A)}=[\int_X \|f(x)\|_A^p w(x)\,d_{\mu(x)}]^{\frac{1}{p}}<\infty\}.$$

Clearly, $L_{p,w}(A)$ is a Banach space [4].

An F-seminorm of a (real) linear space E is a function $\nu: E\rightarrow R_+$ verifying, for $x, y\in E$, $r\in R$,
$\nu(x+y)\leqq\nu(x)+\nu(y)$, $\nu(rx)\leqq\nu(x)$ if $|r|\leqq 1$,
and $\nu(rx)\rightarrow 0$ when $r\rightarrow 0$

An F-seminorm ν is an F-norm when $\nu(x)=0 \Rightarrow x=0$. Let ϕ be a sub-additive Orlicz function, that is increasing and continuous map $\phi: R_+\rightarrow R_+$ verifying $\phi(0)=0$, $\phi(r)>0$ for $r>0$ and $\phi(r_1+r_2)\leq\phi(r_1)+\phi(r_2)$ for $r_1, r_2\in R_+$. If S is a set, we consider the Orlicz space $1^\phi(S)$ of generalized real-sequences $x=(x_s)_{s\in S}$ defined by

$$1^\phi(S)=\{x\in R^S;\ \|x\|_\phi=\sum_{s\in S}\phi(|x_s|)<\infty\}.$$

$1^\phi(S)$ is a complete metrizable topological vector space for the F-norm $\|\cdot\|_\phi$ [6]. We will call a linear space is an F-Banach space if the space is complete with respect to an F-norm. By the above considerations, we have the following

THEOREM 1. *$L_{p,w}(1^\phi(N))$ is an F-Banach space with respect to the F-norm $\|\cdot\|_{L_{p,w}(1^\phi(N))}$.*

In the sequel, we will assume $X=\Omega\subset R^n$ with the Lebesgue measure and denote $1^\phi(N)$ by 1^ϕ. Let $\phi_\alpha(\xi)$, $\xi\in R_+$ are N-functions [1]. Put

$$K_{\phi_\alpha,w}(A)=\{f;\ f(x)\in A,\ \int_\Omega \phi_\alpha(\|f(x)\|_A w(x))\,dx<\infty\}.$$

In general, this class is not a vector space. It is a vector space if ϕ_α satisfies the Δ_2-condition [1]. The space $L_{\phi_\alpha,w}(A)$ is defined to be the linear hull of the class $K_{\phi_\alpha,w}(A)$ $\cdot$ Thus $K_{\phi_\alpha,w}(A)\subset L_{\phi_\alpha,w}(A)$ and these sets are equal if ϕ_α satisfies the Δ_2-condition. $L_{\phi_\alpha,w}(A)$ is a Banach space with respect to the Luxemburg norm [1, 3]

$$\|f\|_{L\phi_\alpha,w(A)}=\inf_k\{k>0;\int_\Omega\phi_\alpha(\frac{1}{k}\|f(x)\|_A w(x))dx\leqq 1\}.$$

Thus we have the following

THEOREM 2. *$L_{\phi_\alpha,w}(1^\phi)$ is an F-Banach space with respect to the F-norm* $\|\cdot\|_{L\phi_\alpha,w(1^\phi)}$.

Proof. Since $\phi(t+t)\leqq 2\phi(t)$ for all $t\in R_+$, for every $r>1$ there exists a positive constant $k=k(r)$ such that for all $t\in R_+$, $\phi(rt)\leq k\phi(t)$. Thus $L_{\phi_\alpha,w}(1^\phi)$ is a linear space for $\phi(rt)\leq\phi(t)$ if $0\leqq r\leqq 1$. The completeness follows from the fact that $L_{\phi_\alpha,w}(A)$ is a Banach space if A is a Banach space.

REMARK. H. Triebel [5] considered $L_{p,w}(l_q)$ space which is the case $\phi=|t|^q$ and $\phi_\alpha(t)=|t|^p$ in our space. Since ϕ_α and ϕ are increasing, we have the following easy imbedding

THEOREM 3.
(i) *If ϕ_α dominates ϕ_β, then* $L_{\phi_\alpha,w}(1^\phi)\longrightarrow L_{\phi_\beta,w}(1^\phi)$.
(ii) *If ϕ dominates ϕ_1, then* $L_{\phi_\alpha,w}(1^\phi)\longrightarrow L_{\phi_\alpha,w}(1^{\phi_1})$.

The general weighted Orlicz-Sobolev spaces $W^m L_{\phi_\alpha,w}(1^\phi)$ is defined by

$$W^m L_{\phi_\alpha,w}(1^\phi)=\{f;D^\alpha f\in L_{\phi_\alpha,w}(1^\phi),\ 0\leqq|\alpha|\leqq m\}$$

with the F-norm, $\|f\|_{W^m L\phi_\alpha,w(1^\phi)}=\sum_{|\alpha|\leqq m}\|D^\alpha f\|_{L\phi_\alpha,w(1^\phi)}$.

THEOREM 4. *$W^m L_{\phi_\alpha,w}(1^\phi)$ is an F-Banach space with respect to the F-norm* $\|\cdot\|_{W^m L\phi_\alpha,w(1^\phi)}$.

Proof. Since $L_{\phi_\alpha,w}(1^\phi)$ is an F-Banach space for every α, $|\alpha|\leqq m$, the linear space $W^m L_{\phi_\alpha,w}(1^\phi)$ is complete with respect to the F-norm by Theorem 7.1.3. of [3].

Let $\mathring{W}^m L_{\phi_\alpha,m}(1^\phi,a_\alpha)=\{f\,|\,D^\alpha f\ \in L_{\phi_\alpha,w}(1^\phi)$ and

$$\|f\|_{\mathring{W}^m L\phi_\alpha,w(1^\phi,a_\alpha)}=\sum_{|\alpha|\leqq m}a_\alpha\|D^\alpha f\|_{L\phi_\alpha,w(1^\phi)}<\infty\}$$

where $a_\alpha>0$.

Then, for finite m, $\mathring{W}^m L_{\phi_\alpha,w}(1^\phi,a_\alpha)=W^m L_{\phi_\alpha,w}(1^\phi)$.

REMARK. The space $\mathring{W}^m\{a_\alpha,p\}$ [2] is the special one of our spaces.

Let

(*) $\quad X_1\supset X_2\supset\cdots\supset X_m\supset\cdots$

be a sequence of compatibly imbedded F-Banach spaces whose F-norms satisfy the inequalities,

$$\|x\|_1\leq\|x\|_2\leq\cdots\leq\|x\|_n\leq\cdots.$$

The limit of the decreasing sequence (*) is, by definition, the space

$$X_\infty=\lim_{m\to\infty} X_m \equiv \{x\in \bigcap_{n=1}^{\infty} X_n ; \|x\|_\infty=\lim_{n\to\infty}\|x\|_n<\infty\}.$$

It is easy to see that X_∞ is an F-Banach space with respect to the F-norm $\|\cdot\|_\infty$. However, it can happen that X_∞ consists only of the element zero, i.e., is trivial. In accordance with definition,

$$\lim_{m\to\infty} \mathring{W}^m L_{\phi_\alpha, w}(1^\phi, a_\alpha)=\mathring{W}^\infty L_{\phi_\alpha, w}(1^\phi, a_\alpha).$$

The criteria of nontriviality of $\mathring{W}^\infty L_p(1^\phi, a_\alpha)$ and $W^\infty L_{\phi_\alpha}(1^\phi)$ where 1^ϕ is the set of scalar valued functions, is considered in [2, 7] where Ω is a domain of finite measure. Since 1^ϕ contains all the scalar valued functions, the criterias of nontriviality of our spaces are Theorem 1, 2 and 3 in [7].

REMARK. 1) $W^m L_{\phi_\alpha, w}(1^\phi)=L_p$, if $m=0$, $w\equiv 1$, $\phi_\alpha(t)=|t|^p$ and 1^ϕ is the set of scalar valued functions.

2) $W^m L_{\phi_\alpha, w}(1^\phi)=L_{\phi_\alpha}$, Orlicz spaces if $m=0$, $w\equiv 1$ and 1^ϕ is the set of scalar valued functions.

3) $W^m L_{\phi_\alpha, w}(1^\phi)=W^m L_{\phi_\alpha}$, Orlicz-Sobolev spaces if $w\equiv 1$ and 1^ϕ is the set of scalar valued functions.

4) $W^m L_{\phi_\alpha, w}(1^\phi)=L_p(l^q)$, l^q-valued L_p-spaces if $m=0$, $\phi_\alpha=|t|^p$ and $\phi=|t|^q$.

5) $W^m L_{\phi_\alpha, w}(1^\phi)=W^m L_{p, w}=W_p{}^m(w)$ weighted sobolev spaces considered in [5], if $\phi_\alpha=|t|^p$ and 1^ϕ is the set of scalar valued functions.

References

1. R. A. Adams, *Sobolev spaces*, Academic Press, 1975.
2. Ju. A. Dubinskii, *Limits of monotone sequences of Banach spaces, applications*, Soviet Math. Dokl., **21** (1980), 465-468.
3. A. Kufner et al, *Function spaces*, Noordhoff International Pub., Leyden, 1977.
4. H. Triebel, *Interpolation theoey, function spaces, differential operators*, North-Holland Pub. Company, 1978.
5. ________, *Maximal inequalities, Fourier multipliers, Littlewood-Paley theorems and approximations by entire analytic functions in weighted spaces*, Analysis Mathematica, **6**(1980), 87-103.
6. P. Turpin, *Properties of Orlicz-Pettis or Nikodim type and barrelledness conditions*, Ann. Inst. Fourier, Grenoble **28**, **3**(1978), 67-85.
7. Chan Dyk Van, *Nontriviality of Sobolev-Orlicz spaces of infinite order in a bounded domain of Euclidean space*, Soviet Math. Dokl., **21** (1980), 335-338.

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