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ABSOLUTE VALUES OF QUASINILPOTENT OPERATORS

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Let l^2 be the Hilbert space of all absolutely square summable sequences $(x_0, x_1, x_2, \cdots)$ of complex numbers. The weighted shift operator S on l^2 is defined by

$$S(x_0, x_1, x_2\cdots) = (0, \lambda_0 x_0, \lambda_1 x_1, \cdots),$$

where $\{\lambda_n : n=0, 1, 2\cdots\}$ is a bounded sequence of complex numbers. It is well known that S is quasinilpotent, if $\lambda_n \to 0$ (p. 101, [1]). The absolute value $|S|$ of S is the diagonal operator with diagonal entries $|\lambda_0|, |\lambda_1|, |\lambda_2|, \cdots$, relative to the standard basis. It follows that zero belongs to the essential spectrum $\sigma_e(|S|)$ of $|S|$.

By borrowing an idea in (p. 101, [1]), we want to prove the following theorem which implies that quasinilpotent operators exist in abundance.

THEOREM. *Let P be a positive (bounded linear) operator on a separable infinite dimensional Hilbert space H such that $0 \in \sigma_e(P)$. Then there is a quasinilpotent operator S on H such that $\sigma_e(|S|) = \sigma_e(P)$. Conversely, if S is a quasinilpotent operator on H, then $0 \in \sigma_e(|S|)$.*

Proof. We only prove the first part of the theorem, since the converse part is easily checked. Let $0 \in \sigma_e(P)$. We first consider the case that 0 is an accumulation point of $\sigma_e(P)$. We can find a sequence $\{\lambda_n\} \subset \sigma_e(P)$ of strictly decreasing sequence of positive real numbers such that $\lambda_n \to 0$. Let $E(\cdot)$ denote the spectral measure of P and $E_n = E([\lambda_n, \lambda_{n-1}))$, $n=1, 2, 3, \cdots$, $E_0 = E([\lambda_0, \infty))$, where $\lambda_0 \leqq \|P\|$. Thus

$H = \sum_{n=0}^{\infty} \oplus H_n$, where $H_n = E_n(H)$. We may assume that

$$\dim(H_n) = \aleph_0 \text{ for all } n=0, 1, 2, 3\cdots.$$

Let $U_0(H_0)=0$. $U_{i+1}: H_{i+1} \to H_i$, $i=0, 1, 2, \cdots$ be an isometric linear surjection, which is possible because $\dim(H_{i+1}) = \dim(H_i)$, $i=0, 1, 2, \cdots$, and put $U = \sum_{i=0}^{\infty} \oplus U_i$. Also, let $P_i = P|H_i$, $i=0, 1, 2, \cdots$, and write $P = \sum_{i=0}^{\infty} \oplus P_i$.

Then, for each

$$\xi = \sum \oplus \{x_n ; n=0, 1, 2\cdots\}, \text{ where } x_n \in H_n \text{ and } k=2, 3\cdots,$$

we have

$$(PU)^k\xi = P_0U_1P_1U_2\cdots P_{k-1}U_kx_k \oplus P_1U_2P_2U_3\cdots P_kU_{k+1}x_{k+1}\oplus\cdots.$$

Put $T=PU$ and $\|\xi\|\leqq 1$. Then,

$$\|T^k\xi\| \leqq \sup\{\|P_0\|\|P_1\|\cdots\|P_k\|, \|P_1\|\|P_2\|\cdots\|P_{k+1}\|, \cdots\}$$
$$\leqq \sup((\|P_0\|+1)\lambda_0\cdots\lambda_{k-1}, \lambda_0\lambda_1\cdots\lambda_k, \cdots\} \leqq (\|P_0\|+1)\lambda_0\cdots\lambda_{k-1}.$$

Thus

$$\|T^k\|^{\frac{1}{k}} \leqq ((\|P_0\|+1)\lambda_0\cdots\lambda_{k-1})^{\frac{1}{k}} \leqq \frac{(\|P_0\|+1)+\lambda_0+\cdots+\lambda_{k-1}}{k}$$
$$\rightarrow 0 \quad (\text{as } k\rightarrow\infty),$$

since $\lambda_{k-1}\rightarrow 0$ (as $k\rightarrow\infty$).
It follows that T is quasinilpotent. We put $S=T^*$, so that $\sigma_e(|S|)=\sigma_e(P)$, while S is also a quasinilpotent operator.

Now we consider the case that $0\in\sigma_e(P)$ and 0 is an isolated point of $\sigma_e(P)$. By the fact that an acumulation point of the spectrum $\sigma(P)$ lies on $\sigma_e(P)$, we clearly see that 0 is an isolated point of $\sigma(P)$ as well. By a result of Stampfli[3], 0 is an eigenvalue of P having an infinite dimensional eigenspace. It follows that $P=0\oplus Q$ with respect to a decomposition $H=H_1\oplus H_2$, where, H_i $(i=1,2)$ are infinite dimensional subspaces and Q is a positive operator on H_2. We put

$$T=\left(\begin{array}{c|c} 0 & Q \\ \hline 0 & 0 \end{array}\right)$$

with respect to $H=H_1\oplus H_2$. Then clearly T is a nilpotent operator with $\sigma_e(|T|)=\sigma_e(P)$. Q. E. D.

References

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