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UNSOLVABILITY OF THE MIZOHATA OPERATOR

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§1. Introduction

Since 1957 when H. Lewy showed that the partial differential equation

$$u_x+iu_y+2(ix-y)u_t=f(x, y, t)$$

is not locally solvable for the generic C^∞-function f in $\mathbf{R}^3$, many other simpler equations with C^∞ coefficients without local solutions have been discovered. One of the simplest forms of this kind is the partial differential equation derived from the Mizohata operator; namely,

$$(*) \qquad Mu=\frac{\partial u}{\partial t}+it\frac{\partial u}{\partial x}=f(x, t).$$

That this is not locally solvable in the space of distributions at any points lying on the x-axis for the generic C^∞-function f can be shown easily following the general criteria, due to Nirenberg and Treves, for the local solvability. (cf. [2], [3], [4])

It seems, however, to us that it is not known yet the complete characterization of those C^∞-functions f for which the equation $(*)$ is not locally solvable, for example at the origin.

Two known results in this line can be found in the short notes [4] by Treves which deserve to be quoted here.

THEOREM A. *Let* $f(x, t)\in C_0^\infty(R^2)$ *have the following properties:*
(a) $f(x, t)=f(x, -t)$ *for all* $(x, t)\in\mathbf{R}^2$;
(b) *supp* f (i. e., *the support of* f) *does not intersect the axis* $t=0$;
(c) $\iint_{R^2} f(x, t)\,dx\,dt\neq 0$
Then the equation $(*)$ *in* $\mathbf{R}^2$

$$Mu=f$$

does not have any solution in $\mathcal{D}'(\mathbf{R}^2)$.

THEOREM B. *Let* $f(x, y)\in C_0^\infty(\mathbf{R}^2)$ *be such that*

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$$Kf(x)=\frac{1}{2\pi}\int_{t=-\infty}^{+\infty}\int_{\xi=0}^{+\infty}\{\exp(ix\xi-\frac{t^2|\xi|}{2})\}f(x,\xi)\,dt\,d\xi$$

is an analytic function of x in $\mathbf{R}^1$.

Then the equation (*) in $\mathbf{R}^2$ has a C^1-solution in a neighborhood of any point in $\mathbf{R}^2$.

In particular, if $f(x,t)$ is odd with respect to t, the equation (*) is always locally solvable at any point in $\mathbf{R}^2$.

The aim of this article is to generalize the Theorem A quoted above and to get similar result *without* assuming the condition (b) that *supp* f does not intersect the axis $t=0$.

2. Theorem

THEOREM. *Let $f(x,t)\in C_0^\infty(\mathbf{R}^2)$ have the following properties:*

(1) *$f(x,t)=f(x,-t)$ for all $(x,t)\in R^2$;*

(2) *there exists a sequence $\{K_n\}$ $(n=1,2,3\cdots)$ of mutually disjoint compact subset of $\mathbf{R}^2$ such that*

(i) *$K_n\subset\{(x,t)\,|\,t\geqq c|x|\}$ $(n=1,2,3,\cdots)$ for some fixed constant $c>0$,*

(ii) *$\lim_{n\to\infty}K_n=\{0\}$ where 0 is the origin, and*

(iii) *$supp\ f\subset\bigcup_{n=1}^{\infty}(K_n\cup K_n^-)$, where $K_n^-=\{(x,-t)\,|\,(x,t)\in K_n\}$;*

(3) *$\iint_{R^2}f(x,t)\,dxdt\neq0$.*

Then the equation in $\mathbf{R}^2$

$$(*)\qquad Mu=\frac{\partial u}{\partial t}+it\frac{\partial u}{\partial x}=f(x,t)$$

does not have any solution in $\mathcal{D}'(\mathbf{R}^2)$.

Proof. We shall prove that the equation (*) does not have any solution u which is C^1-function of t valued in $\mathcal{D}_x'$, since all solutions of (*) have necessarily this property. (*cf.* [4])

By (i), setting $s=\frac{1}{2}t^2$, we may write

$$f(x,t)=F(x,s)\qquad(s\geqq0).$$

We shall then define $F(x,s)=0$ for $s<0$, Thus $F(x,s)$ is a function defined on R^2.

Suppose now that there exists a solution u of (*), C^1 with respect to t valued in $\mathcal{D}_x'$. Since u can be decomposed as a sum of even and odd terms with respect to t-variable, we have

$$u(x,t)=v(x,s)+tw(x,s)\qquad(s\geqq0).$$

Since $\frac{ds}{dt}=t$, we may set, for $s>0$,

$$Mu=\left(\frac{\partial}{\partial t}+it\frac{\partial}{\partial x}\right)(v(x,s)+tw(x,s))$$
$$=tv_s+w+2sw_s+itv_x+2isw_x$$
$$=t(v_s+iv_x)+(w+2sw_s+2isw_x)$$

Therefore, for $s>0$,

$$t(v_s+iv_x)+(w+2s\ w_s+2isw_x)=F(x,s)$$

or, equivalently, for $s>0$,

$$v_s+iv_x=0, \tag{4}$$
$$w+2sw_s+2isw_x=F(x,s) \tag{5}$$

which follows from the fact that $F(x,s)=f(w,t)$ is even with respect to t.

Now the equation (5) can be written as

$$\frac{1}{2\sqrt{s}}w+\sqrt{s}\,w_s+i\sqrt{s}\,w_x=\frac{F}{2\sqrt{s}} \qquad (s>0)$$

or,

$$(\sqrt{s}\,w)_s+i(\sqrt{s}\,w)_x=\frac{F}{2\sqrt{s}} \qquad (s>0). \tag{6}$$

We set here $\sqrt{s}\,w(x,s)=h(z)$ where $z=s+ix$ for $s\geqq 0$, $x\neq 0$, and consider the set Q, the complement of *supp* F in the half space $s>0$. The equation (6) shows that $h(z)$ is holomorphic in Q, since in Q

$$\frac{\partial}{\partial\bar{z}}h(z)=\frac{1}{2}\left(\frac{\partial}{\partial s}+i\frac{\partial}{\partial x}\right)(\sqrt{s}\,w)=0.$$

We recall that u is assumed to be C^1 function with respect to t-variable. This implies that $w(x,s)$ and hence $h(z)=\sqrt{s}\,w(x,s)$ is a continuous function on $Q\cup\{z=s+ix\,|\,s=0,\ x\neq 0\}$. Moreover, as $\sqrt{s}\,w(x,s)$ vanishes when $s=0$, $h(z)$ takes the purely imaginary value (in fact, 0) on the imaginary axis $s=0$, with the origin excluded. Therefore, by the reflection principle, $h(z)$ can be extended as a holomorphic function, say $h(z)$ again, to V, the unbounded connected component of $\mathbf{R}^2/\{(supp/F)\ \cup (supp\ F)^-\}$ where, as usual,

$$(supp\ F)^-=\{(x,-t)\,|\,(x,t)\in\ supp\ F\}.$$

Since $h\equiv 0$ on the imaginary axis (with origin excluded), it follows that

$$h(z)\equiv 0 \text{ identically on } V.$$

In particular,

$$\sqrt{s}\,w(x,s)\equiv 0 \text{ identically on } V\cap\{(x,s)\,|\,s\geqq 0\}.$$

We extend $w(x,s)$ as a function on $\mathbf{R}^2$ such that

$$w(x,s)=0 \text{ if } s<0$$

(Then $\sqrt{s}\,w(x,s)$ is a solution of the hypoelliptic partial differential equation $\frac{\partial u}{\partial s}+i\frac{\partial u}{\partial x}=\frac{F}{2\sqrt{s}}$ in $\mathbf{R}^2/\{0\}$.)

Now we note that $supp\ F\subset\bigcup_{n=1}^{\infty}G_n$ where

$$G_n=\{(x,s)\,|\,s=\frac{1}{2}t^2,\ (x,t)\in K_n\}.$$

Since K_n's are also mutually disjoint, G_n's are also mutually disjoint.

Therefore we can find a large circle Γ with center at the origin enclosing all the G_m's $(m=1,2,3,\cdots)$ and a small contour Γ_n enclosing all G_k's for $k\geqq n$ and such that $\Gamma_n\cap\bigcup_{m=1}^{\infty}G_m=\phi$. Let D_n be the region surrounded by Γ and Γ_n.

Now let us notice that $F(x,s)=0$ in a neighborhood of $\{(x,s)\,|\,s=0,\ x\neq 0\}$. Therefore, if we set

$$k(x,s)=\begin{cases}\frac{F(x,s)}{\sqrt{s}} & \text{if } s\neq 0,\\ 0 & \text{if } s=0,\end{cases}$$

then, by the fact that $f(x,t)=F(x,s)$ $(s\geqq 0)$, $s=\frac{1}{2}t^2$ and from the symmetry of f with respect to t, we have

$$I=\iint_{R^2}f(x,t)dx\ dt=\lim_{n\to\infty}\iint_{D_n}f(x,t)dx\ dt=2\lim_{n\to\infty}\iint_{D_n}k(x,s)dx\ ds.$$

Since $\sqrt{s}\,w(x,s)$ is a solution of the hypoelliptic partial differential equation

$$\frac{\partial u}{\partial s}+i\frac{\partial u}{\partial s}=\frac{k(x,s)}{2}$$

in $\mathbf{R}^2/\{0\}$ and $k(x,s)$ is C^∞ in $\mathbf{R}^n/\{0\}$, it follows that $\sqrt{s}\ w(x,s)$ is C^∞-function in $\mathbf{R}^n/\{0\}$. Therefore we have, by the Stoke's theorem, that

$$\begin{aligned}I&=4\lim_{n\to\infty}\iint_{D_n}\{(\sqrt{s}\,w)_s+i(\sqrt{s}\,w)_x\}\,dx\ ds\\&=4\lim_{\to\infty}\left[-\iint_{D_n}\frac{\partial(\sqrt{s}\,w)}{\partial s}ds\ dx+\iint_{D_n}\frac{\partial(i\sqrt{s}\,w)}{\partial x}dx\ ds\right]\\&=4\lim_{\to\infty}[-\int_{\Gamma}\sqrt{s}\,wdx+\int_{\Gamma}i\sqrt{s}\,wds+\int_{\Gamma_n}\sqrt{s}\,wdx-\int_{\Gamma_n}i\sqrt{s}\,wdx]\end{aligned}$$

But as $\sqrt{s}\ w$ vanishes on Γ_n and Δ, we have

$I=\iint_{R^2}f(x,t)dxdt=0$, contrary to our hypothesis.

This completes our proof.

REMARK. We remark that, in general, *supp* f may intersect with x-axis at the origin.

References

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