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SEMI-WEAKLY DECOMPOSABLE AND SEMI-ANALYTICALLY DECOMPOSABLE OPERATORS

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1. Introduction

Throughout this note, T is a bounded linear operator on a complex Banach space X. An invariant subspace Y of T is called a spectral maximal subspace if Y contains all invariant subspace M for which $\sigma(T|M)\subset\sigma(T|Y)$. The operator T is called *decomposable* if for every finite open cover $\{G_1, G_2, \cdots G_n\}$ of $\sigma(T)$ there are invariant subspaces $Y_1, Y_2, \cdots Y_n$ such that

(1) Y_i is a spectral maximal subspace for each i,

(2) $\sigma(T|Y_i)\subset G_i$ $(i=1, 2, \cdots, n)$,

(3) $X=Y_1+Y_2+\cdots+Y_n$.

The operator T is *weakly decomposable* if we replace the condition

(3′) $X=\bigvee_{i=1}^{n} Y_i$(closed linear span of $Y_1, Y_2, \cdots Y_n$) instead of (3).

An invariant subspace Y of T is said to be *analytically invariant* if for each X-valued analytic function f defined on V_f in C such that $(\lambda-T)f(\lambda)\in Y$ for $\lambda\in V_f$, then $f(\lambda)\in V_f$ for $\lambda\in V_f$.

A bounded linear operator T is said to be *analytically decomposable* if for any finite open cover $\{G_1, G_2, \cdots, G_n\}$ of $\sigma(T)$, there are invariant subspaces $Y_i (i=1, 2, \cdots, n)$ such that

(i) Y_i is analytically invariant (ii) $\sigma(T|Y_i)\subset G_i$ for each i

(iii) $X=\bigvee_{i=1}^{n} Y_i$.

2. Spectrum of a weakly decomposable operator

It is known that an operator T is decomposable$\Rightarrow$Weakly decomposable $\Rightarrow$ analytically decomposable; the first implication is obvious, the second is true since every spectral maximal subspace of T is analytically invariant but not the converse in general (see[3]).

An open question is that whether or not the second implication is reversible, we will give a partial answer of this question in proposition 2. 3

below. A complete answer will be given for the semi-analytically decomposable operator and the semi-weakly decomposable operator. (see Theorem 3. 3)

By definitions, each weakly decomposable operator is analytically decomposable. So we have the following facts:

(a) A weakly decomposable operator has the single valued extension property,

(b) If T is weakly decomposable, then $\sigma(T)=\sigma_{ap}(T)$, where $\sigma_{ap}(T)$ is the approximate point spectrum of T.

Proofs for the analytically decomposable operator are given in [3].

If Y is an ultra-invariant subspace of T, then it is known that $\sigma(T)=\sigma(T|Y)\cup\sigma(T^Y)$, where $T^Y\in B(X/Y)$ is the quotient operator induced by T (see[1], Lemma 3. 1, p. 1487). Since every spectral maximal subspace of T is ultra-invariant, the above equality holds for any spectral maximal subspace of T.

It is true that $\sigma_{ap}(T|Y)\subset\sigma_{ap}(T)$, but in general there is no inclusion relation $\sigma_{ap}(T^Y)$ and $\sigma_{ap}(T)$ in spite of $\sigma(T^Y)\subset\sigma(T)$. Furthermore, if T is weakly decomposable, we do not know whether or not $T|Y$, T^Y are weakly decomposable even if Y is a spectral maximal subspace of T. Therefore, we are unable to say the equality $\sigma_{ap}(T^Y)=\sigma(T^Y)$ or $\sigma_{ap}(T|Y)=\sigma(T|Y)$ hold. We may prove, however, the following proposition:

2. 1. PROPOSITION *Let T be weakly decomposable and Y a spectral maximal subspace of T, then*

$$\sigma_{ap}(T)=\sigma(T|Y)\cup\sigma_{ap}(T^Y).$$

Proof. It is known that $\sigma(T|Y)\subset\sigma(T)$ holds for any spectral maximal subspace of T. For any $\lambda\in\sigma(T)\backslash\sigma(T/Y)$, since $\sigma(T)=\sigma_{ap}(T)$, there exists a sequence $\{x_n\}$ such that $\|x_n\|=1$ for each n and $(\lambda I-T)x_n\to 0$ $(n\to\infty)$. We claim $x_n\notin Y$ for infinitely many but finite number of n; if $x_n\in Y$ for infinitely many n. then $(\lambda I-T|Y)x_n=(\lambda I-T)y_n\to 0$ as $n\to\infty$, whence $\lambda\in\sigma_{ap}(T|Y)\subset\sigma(T|Y)$, a contradiction.

Furthermore, if there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_k x_{n_k}=x_0$ exists in Y, then $\lim_k(\lambda I-T)x_{n_k}=(\lambda I-T|Y)x_0=0$. $(x_0\neq 0)$. Thus λ is an eigenvalue of $T|Y$, therefore $\lambda\in\sigma(T|Y)$, a contradiction. Thus without loss of generality we may assume that $x_n\notin Y$ for every n, and there is no subsequence of $\{x_n\}$ such that the limit exist in Y. It follows that $\|x_n'\|=\inf\{\|x_n+y\|:y\in Y\}\neq 0$ for any x_n and $\{x_n'\}$ does not converges to $0'=Y$, where $x'=x+Y$.

Hence $0<\|x_n'\|\leq\|x_n\|=1$ for each n, x_n' does not converge to $0'$.

Now we put

$$z_n' = x_n'/\|x_n'\| \text{ for each } n,$$

then $z_n' \in X/Y$, $\|z_n'\|=1$ for each n.
Since $(\lambda I - T^Y)z_n' = (\lambda I - T)x_n/\|z_n'\| + Y \to 0' = Y$, we have $\lambda \in \sigma_{ap}(T^Y)$.
It follows that

$$\sigma(T)\backslash\sigma(T|Y) \subset \sigma_{ap}(T^Y),\ \sigma(T) \subset \sigma(T|Y) \cup \sigma_{ap}(T^Y) \subset \sigma(T),$$

i.e., $\sigma_{ap}(T) = \sigma(T|Y) \cup \sigma_{ap}(T^Y)$.

2.2. LEMMA *Let Y be an invariant subspace of T, then $\sigma(T^Y) \cap \sigma(T|Y) = \phi$. if and only if $\sigma(x' T^Y) \subset \sigma(x, T)\backslash\sigma(T|Y)$ for each $x \in X$, where $x' = x + Y$.*

Proof. Suppose $\sigma(T^Y) \cap \sigma(T|Y) = \phi$. Since $\bigcup_{x' \in X/Y} \sigma(x', T^Y) = \sigma(T^Y)$ we have $\sigma(x', T^Y) \cap \sigma(T|Y) = \phi$ for each x'.

It is easily be shown that $\sigma(x', T^Y) \subset \sigma(x, T)$; for, each $\lambda \in \rho(x, T)$, there exists an analytic X-valued function f such that

$$(\lambda - T)f(\lambda) = x,$$

whence $(\lambda - T)^Y[f(\lambda)]' = x'$, where $[f(\cdot)]' : (x', T^Y) \to X/Y$ is an analytic function. Hence $\rho(x, T) \subseteq \rho(x', T^Y)$. Therefore, we have $\sigma(x', T^Y) \subset \sigma(x, T)\backslash\sigma(T|Y)$.

Conversely, if $\sigma(x, T^Y) \subset \sigma(x, T)\backslash\sigma(T|Y)$ for each $x \in X$, then $\sigma(T^Y) \subset \sigma(T)\backslash\sigma(T|Y)$, thus $\sigma(T^Y) \cap \sigma(T|Y) = \phi$.

2.3. PROPOSITION *Let T be analytically decomposable. If $\sigma(T)$ is the disjoint union of $\sigma(T|Y)$ and $\sigma(T^Y)$ for any analytically invariant subspaces Y of T, then T is weakly decomposable.*

Proof. It is known that if Y is an analytically invariant subspace, then $\sigma(T) = \sigma(T|Y) \cup \sigma(T^Y)$. It is enough to show the assumption implies that Y is a spectral maximal subspace of T. Suppose Z is invariant under T such that $\sigma(T|Z) \subset \sigma(T|Y)$. If $x \in Z$, then

$$\sigma(x, T) \subset \sigma(T|Z) \cup \sigma(T|Y).$$

Thus, by Lemma 2.2, we have

$$\sigma(x', T^Y) \subset \sigma(x, T)\backslash\sigma(T|Y) = \phi,$$

therefore

$$x' = x + Y = Y \text{ or } x \in Y, \text{ whence } Z \subset Y.$$

3. Semi-analytically and semi-weakly decomposable operators

3.1. DEFINITION An operator T is said to be *semi-analytically decomposable* if any finite open covering $\{G_i\}_{i=1}^n$ of $\sigma(T)$ there are corresponding a system of analytically invariant subspaces $\{Y_i\}_{i=1}^n$ of T such that

(i) $\sigma(T/Y_i) \subset G_i\ (i=1, 2, \cdots n)$,

(ii) there exists at least one $Y_k\ (1 \le k \le n)$ such that $X = Y_k + \bigvee_{i \neq k} Y_i$

3.2. DEFINITION An operator T is called *semi-weakly decomposable* if we replace spectral maximal subspaces instead of analytically invariant subspaces in Definition 3.1.

By definition, a semi-weakly decomposable operator is two-decomposable thus it is decomposable (see[4]). Thus the notion of semi-weakly decomposable operator is same as the decomposable operator. Also we can say that a semi-weakly decomposable operator is semi-analytically decomposable since every spectral maximal subspace is analytically invariant. Now, we shall show the converse is valid:

3.3. THEOREM *T is semi-analytically decomposable if and only if T is semiweakly decomposable.*

For the proof of theorem 3.3, we begin with the following

3.4. LEMMA *Let T be a semi-analytically decomposable operator. For every closed set F in $\mathbf{C}$ (or $\sigma(T)$), $X_T(F)=\{x\in X:\sigma(x, T)\subset F\}$ is closed in X. Thus $X_T(F)$ is a spectral maximal subspace of T.*

Proof. Since T is semi-analytically decomposable, T is analytically decomposable. Therefore T has the single valued extension property, so $X_T(F)$ is defined. For any open covering $\{G_1, G_2\}$ of $\sigma(T)$, there exist analytically invariant subspaces Y_1, Y_2 such that

$$\sigma(T|Y_i)\subset G_i \ (i=1,2) \text{ with } Y_1+Y_2=X.$$

Therefore by the same calculation as in the proof of ([2], Theorem 1.5, p. 31), $X_T(F)$ is closed in X.

Proof of Theorem 3.3. Let $\{G_i\}_{i=1}^n$ be any finite open covering of $\sigma(T)$. We have to seek a system of spectral maximal subspaces $\{Z_i\}_{i=1}^n$ such that

$$\sigma(T|Z_i)\subset G_i \text{ for each } i \text{ and } Z_k+\bigvee_{i\neq k} Z_i=X.$$

We consider another open covering $\{H_i\}_{i=1}^n$ of $\sigma(T)$ such that $\bar{H}_i\subset G_i$ for each i. By definition, there are corresponding analytically invariant subspaces $\{Y_i\}_{i=1}^n$ such that

$$(1)\quad \sigma(T|Y_i)\subset H_i \text{ for each } i,\quad Y_k+\bigvee_{i\neq k} Y_i=X.$$

Since $\sigma(y, T)\subseteq\sigma(T|Y_i)\subset\bar{H}_i$ for each $y\in Y_i$, $y\in X_T(\bar{H}_i)$.
Thus $Y_i\subset X_T(\bar{H}_i)$ for each i.
And $X_T(\bar{H}_i)=X_T(\bar{H}_i\cap\sigma(T))=X_T(F_i)$, where $F_i=\bar{H}_i\cap\sigma(T)$,
is closed for each i by Lemma 3.4. Therefore we have a system of spectral maximal subspaces $\{X_T(F_i)\}_{i=1}^n$ of T such that

$$Y_k + \bigvee_{i \neq k} Y_i \subset X_T(F_k) + \bigvee_{i \neq k} X_T(F_i).$$

Hence we have $X = X_T(F_k) + \bigvee_{i \neq k} X_T(F_i)$.

Furthermore, since $\sigma(T|X_T(F_i)) \subseteq F_i \subset \bar{H}_i \subset G_i$ for each i, if we put $X_T(F_i) = Z_i$ for each i, we have the required spectral maximal subspaces of T.

The following corollary is immediate consequence of the Theorem 3.3:

3.5. COROLLARY *If T is semi-analytically decomposable, then T is weakly decomposable.*

3.6. COROLLARY *T is semi-analytically decomposable if and only if T is decomposable.*

For, T is semi-analytically decomposable if and only if T is semi-weakly decomposable if and only if T is decomposable.

3.7. COROLLARY *If T is semi-analytically decomposable and if f is any non-constant scalar valued analytic function on some neighborhood of $\sigma(T)$, then $f(T)$ is weakly decomposable.*

For, if T is semi-analytically decomposable then T is decomposable by corollary 3.6. And $f(T)$ is weakly decomposable.

It is also obvious that if T is semi-weakly decomposable then $f(T)$ is weakly decomposable.

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