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ON THE ASYMPTOTIC BEHAVIOR OF NONEXPANSIVE MAPS IN BANACH SPACES

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Let E be a Banach space, C a closed convex subset of E, $T:C\to C$ a nonexpansive map, and $F(T)$ the set of fixed points of T. If E is uniformly convex, $F(T)\neq\phi$ and T is asymptotically regular at $x\in C$ (that is, $\lim_n\|T^nx-T^{n+1}x\|=0$), it remains an open question whether $\{T^nx\}$ converges weakly to a fixed point of T. Partial answers in the affirmative were given by Opial [3] for those E that have a weakly sequentially continuous duality map, and by Baillon, Bruck, and Reich [1] for odd T and $C=-C$.

In this paper, we improve the result in [1] by removing the condition $C=-C$ and the convexity and by assuming that T is continuous and satisfies

$$\|Tx+Ty\|\leqslant\|x+y\| \qquad (*)$$

for all x, y in C. Note that if $C=-C$, then T is odd and nonexpansive if and only if (*) holds.

THEOREM. *Let E be a uniformly convex Banach space, C a closed subset of E, and T continuous selfmap of C satisfying (*). If T is asymptotically regular at $x\in C$, then $\{T^nx\}$ converges strongly to a fixed point of T.*

Proof. Since T satisfies (*), $\lim_n\|T^nx\|=d$ exists and $\{\|T^{n+i}x+T^nx\|\}$ is nonincreasing for each i. Since $2d\leqslant 2\|T^nx\|\leqslant\|T^{n+i}x+T^nx\|+\|T^nx-T^{n+i}x\|$ and $\lim_n \|T^nx-T^{n+i}x\|=0$ by the asymptotic regularity, we have $2d\leqslant\|T^{n+i}x+T^nx\|$ for all n and i. Now we have $\lim_n\|T^nx\|=d$ and $\lim_{m,n}\|T^nx+T^mx\|=2d$. By uniform convexity, $\lim_{m,n}\|T^nx-T^mx\|=0$, whence $\{T^nx\}$ converges strongly to some $q\in C$. Since T is continuous, we have $q=Tq$.

Our theorem improves Theorem 1.1 of [1]. Simple examples showing that our improvement is proper are easily constructed. Note that in Theorem 1.1 of [1] the convexity of C can be replaced by the weaker condition $O\in C$. Therefore, if $O\in F(T)$ and $C\neq -C$, then by defining $T(-x)=-Tx$ for $x\in C$, T can be extended to a selfmap of $C\cup -C$, and our theorem may follow from Theorem 1.1 of [1].

Corollaries 2.1, 2.4, Theorems 3.1, 4.1, and Corollary 4.1 of [1] can be also improved in the similar way. Note that Corollary 1.2 in [2] also follows from our theorem.

References

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