Bull. Korean Mtah. Soc.
Vol. 17, No. 2, 1981

A CHARACTERIZATION OF SASAKIAN MANIFOLDS WITH VANISHING C-BOCHNER CURVATURE TENSOR

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Recently, in his papers [5] and [6], J. S. Pak proved the following theorems.

THEOREM A ([5]). *Let M^n be a Sasakian manifold of dimension n with constant scalar curvature whose C-Bochner curvature tensor vanishes. If Ricci tensor is positive semi-definite, then M^n is locally C-Fubinian.*

THEOREM B ([6]). *Let M^n be a Sasakian manifold of dimension n with constant scalar curvature K whose C-Bochner curvature tensor vanishes. If the square of the length of Ricci tensor is less than K^2/n-1, then M^n is locally C-Fubinian.*

THEOREM C ([6]). *Let M^n be a compact Sasakian manifold of dimension n whose C-Bochner curvature tensor vanishes. If the square of the length of Ricci tensor is constant and less than K^2/n-1, then M^n is locally C-Fubinian.*

As an improvement of Theorem C, in their paper [7], J. S. Pak and J. W. Kwon proved

THEOREM D ([7]). *Let M^n be a compact Sasakian manifold of dimension n whose C-Bochner curvature tensor vanishes. If the eigenvalues of Ricci tensor corresponding to eigebvectors contained in the $(n-1)$-dimensional distribution orthogonal to the Sasakian structure vector η are not greater than -4, and if the square of the length of Ricci tensor is not greater than $K^2/n-1$, then M^n is locally C-Fubinian.*

In this paper we shall study Sasakian manifold with vanishing C-Bochner curvature tensor and improve Theorem B and Theorem D.

1. Preliminaries

In 1969, Matsumoto and Chūman [1] introduced a tensor field of type (1, 3) on an n-dimensional Sasakian manifold M^n whose components $B_{kji}{}^h$ are given by

$$(1.1)\quad B_{kji}{}^h = K_{kji}{}^h + \frac{1}{n+3}(K_{ki}\delta_j^h - K_{ji}\delta_k^h + g_{ki}K_j^h - g_{ji}K_k^h + S_{ki}f_j^h - S_{ji}f_k^h$$
$$+ f_{ki}S_j^h - f_{ji}S_k^h + 2S_{kj}f_i^h + 2f_{kj}S_i^h - K_{ki}\eta_j\eta^h + K_{ji}\eta_k\eta^h$$
$$- \eta_k\eta_i K_j^h + \eta_j\eta_i K_k^h) - \frac{k+n-1}{n+3}(f_{ki}f_j^h - f_{ji}f_k^h + 2f_{kj}f_i^h)$$
$$- \frac{k-4}{n+3}(g_{ki}\delta_j^h - g_{ji}\delta_k^h) + \frac{k}{n+3}(g_{ki}\eta_j\eta^h + \eta_k\eta_i\delta_j^h - g_{ji}\eta_k\eta^h - \eta_j\eta_i\delta_k^h),$$

the aggregate $(f_j^i, \eta_j, \eta^i, g_{ji})$ being the Sasakian structure, where $K_{kji}{}^h, K_{ji}$ and K are curvature tensor, Ricci tensor and scalar curvature of M^n respectively, and $S_{kj} = f_k^h K_{hj}$, $S_k^i = S_{kj}g^{ji}$, $(g^{ji}) = (g_{ji})^{-1}$ and $k = (K+n-1)/(n+1)$. They called it C-Bochner curvature tensor and obtained the following identities concerning with this tensor field:

$$(1.2)\quad \begin{aligned} &B_{kji}{}^h = -B_{jki}{}^h, \qquad B_{kjih} = B_{ihkj}, \\ &B_{kji}{}^h + B_{jik}{}^h + B_{ikj}{}^h = 0, \quad B_{kij}{}^k = 0, \\ &B_{kji}{}^h\eta_h = 0, \quad f_k^s B_{sji}{}^h = f_j^s B_{ski}{}^h, \quad f^{kj}B_{kji}{}^h = 0, \end{aligned}$$

where $B_{kjih} = B_{kji}{}^s g_{sh}$, $f^{kj} = f_s^j g^{sk}$.

Here we introduce a tensor field of type $(1,3)$ on M^n whose components $U_{kji}{}^h$ are defined by

$$U_{kji}{}^h = K_{kji}{}^h - (\rho+1)(g_{ji}\delta_k^h - g_{ki}\delta_j^h)$$
$$- (g_{ki}\eta_j\eta^h + \eta_k\eta_i\delta_j^h - g_{ji}\eta_k\eta^h - \eta_j\eta_i\delta_k^h + f_{ji}f_k^h - f_{ki}f_j^h - 2f_{kj}f_i^h),$$
$$\rho + 1 = \frac{k}{n-1},$$

which is an analogy of the concircular curvature tensor in a Kaehlerian manifold. A Sasakian manifold M^n is called locally C-Fubinian [8] when the tensor field $U_{kji}{}^h$ vanishes identically on M^n. If a Sasakian manifold M^n is locally C-Fubinian, its Ricci tensor satisfies

$$K_{ji} = ag_{ji} + b\eta_j\eta_i,$$

where $a = \frac{K}{n-1} - 1$ and $b = -\frac{K}{n-1} + n$. In this case the manifold M^n is said to be C-Einstein [4]. Hence if a Sasakian manifold is locally C-Fubinian, then it is C-Einstein. Using this relation, Matsumoto and Chūman [1] proved

THEOREM E ([1]). *C-Bochner curvature tensor $B_{kji}{}^h$ coincides with $U_{kji}{}^h$ if and only if M^n is C-Einstein space.*

By means of this theorem a Sasakian manifold M^n with vanishing C-Bochner curvature tensor is locally C-Fubinian if M^n is C-Einstein.

2. Fundamental propertites of Sasakian manifolds with vanishing C-Bochner curvature tensor

Let M^n be an n-dimensional Sasakian manifold ($n \geqq 3$). Then we can easily verify that the following equations are established on M^n (*cf.* [4]):

$$\nabla_k S_{ji} = \eta_j K_{ki} - (n-1) g_{kj}\eta_i + f_j{}^t \nabla_k K_{ti},$$
$$(2.1) \qquad \nabla_k S_j{}^k = \frac{1}{2} f_j{}^k \nabla_k K + (K-n+1)\eta_j,$$
$$f_j{}^t \nabla_t S_{ik} = -\eta_i S_{kj} + (n-1) f_{ij}\eta_k + f_j{}^r f_i{}^s \nabla_r K_{sk},$$
$$S_{ji} = -S_{ij}$$

with the help of

$$(2.2) \qquad K_{ji}\eta^i = (n-1)\eta_j.$$

On the other hand, the differential form $S = \frac{1}{2} S_{ji} dx^j \wedge dx^i$ is closed, i. e.,

$$\nabla_k S_{ji} + \nabla_j S_{ik} + \nabla_i S_{kj} = 0,$$

from which and (2. 1), we also obtain

$$(2.3) \qquad \nabla_k K_{ji} - \nabla_j K_{ki} = -f_i{}^r \nabla_r S_{kj} - 2S_{kj}\eta_i + (n-1)(f_{ki}\eta_j - f_{ji}\eta_k + 2f_{kj}\eta_i).$$

We now differentiate (1. 1) covariantly along M^n. Then, by (2. 1), we have

$$(2.4) \quad (n+3)\nabla_t B_{kji}{}^t = (n+2)(\nabla_k K_{ji} - \nabla_j K_{ki}) - f_k{}^r f_j{}^s (\nabla_r K_{si} - \nabla_s K_{ri})$$
$$+ 2 f_i{}^s f_k{}^r \nabla_s K_{rj} + \eta^r (\eta_k \nabla_r K_{ji} - \eta_j \nabla_r K_{ki}) - (n+2)\eta_k S_{ji} + n\eta_j S_{ki}$$
$$+ 2(n+1)\eta_i S_{kj} + \frac{1}{n+1}(g_{ki}\eta_j - g_{ji}\eta_k)\eta^r \nabla_r K + \frac{n-1}{2(n+1)}((g_{ki} -$$
$$\eta_k\eta_i)\nabla_j K - (g_{ji} - \eta_j\eta_i)\nabla_k K + (f_{ki} f_j{}^r - f_{ji} f_k{}^r + 2 f_{kj} f_i{}^r)\nabla_r K)$$
$$+ (n-1)((n+2)\eta_k f_{ji} - n\eta_j f_{ki} - 2(n+1)\eta_i f_{kj}).$$

Transvecting (2. 4) with $f_a{}^k f_b{}^j$ and changing the indices a, b to j, k respectively in the equation thus obtained, we find by adding the resulting to (2. 4)

$$\nabla_t B_{kji}{}^t + f_j{}^r f_k{}^s \nabla_t B_{rsi}{}^t = (\nabla_k K_{ji} - \nabla_j K_{ki}) - f_k{}^r f_j{}^s (\nabla_r K_{si} - \nabla_s K_{ri})$$
$$+ (n-1)(\eta_k f_{ji} - \eta_j f_{ki}) - \eta_k S_{ji} + \eta_j S_{ki} + \frac{1}{2(n+3)}(g_{ki}\eta_j - g_{ji}\eta_k)\eta^t \nabla_t K.$$

On the other hand, using (1. 2), we have

$$f_j{}^r f_k{}^s \nabla_t B_{rsi}{}^t = -\nabla_t B_{kji}{}^t,$$

which together with the last equation imply

$$(2.5) \qquad \nabla_k K_{ji} - \nabla_j K_{ki} - f_k{}^r f_j{}^s (\nabla_r K_{si} - \nabla_s K_{ri}) - \eta_k S_{ji} + \eta_j S_{ki}$$

$$+\frac{1}{2(n+3)}(g_{ki}\eta_j-g_{ji}\eta_k)\eta^r\nabla_r K+(n-1)(\eta_k f_{ji}-\eta_j f_{ki})=0.$$

Transvecting (2.5) with η^k and $\eta^k g^{ji}$ respectively, we can see that

$$\eta^t\nabla_t K=0,\quad \eta^t\nabla_t K_{ji}=0. \tag{2.6}$$

Substituting (2.5) and (2.6) in (2.4), we find

$$\frac{n+3}{n-1}\nabla_t B_{kji}{}^t=\nabla_k K_{ji}-\nabla_j K_{ki}-\eta_k(S_{ji}-(n-1)f_{ji})+\eta_j(S_{ki}-(n-1)f_{ki})$$
$$+2\eta_i(S_{kj}-(n-1)f_{kj})+\frac{1}{2(n+1)}((g_{ki}-\eta_k\eta_i)\delta_j^t-(g_{ji}-\eta_j\eta_i)\delta_k^t$$
$$+f_{ki}f_j{}^t-f_{ji}f_k{}^t+2f_{kj}f_i{}^t)\nabla_t K.$$

Thus in a Sasakian manifold with vanishing C-Bochner curvature tensor, we have

$$\nabla_k K_{ji}-\nabla_j K_{ki}=\eta_k(S_{ji}-(n-1)f_{ji})-\eta_j(S_{ki}-(n-1)f_{ki})$$
$$-2\eta_i(S_{kj}-(n-1)f_{kj})-\frac{1}{2(n+1)}((g_{ki}-\eta_k\eta_i)\delta_j^t$$
$$-(g_{ji}-\eta_j\eta_i)\delta_k^t+f_{ki}f_j{}^t-f_{ji}f_k{}^t+2f_{kj}f_i{}^t)\nabla_t K, \tag{2.7}$$

which together with (2.1), (2.3) and (2.6) yield

$$\nabla_k S_{ji}=\eta_j K_{kj}-\eta_i K_{kj}+\frac{1}{2(n+1)}(f_{jk}\delta_i^t-f_{ik}\delta_j^t+2f_{ji}\delta_k^t+(g_{ik}-\eta_i\eta_k)f_j{}^t$$
$$-(g_{jk}-\eta_j\eta_k)f_i{}^t)\nabla_t K. \tag{2.8}$$

Moreover, comparing (2.1) with (2.8) gives

$$f_j{}^t\nabla_k K_{ti}=(n-1)\eta_i g_{kj}-\eta_i K_{kj}+\frac{1}{2(n+1)}(f_{jk}\delta_i^t-f_{ik}\delta_j^t+2f_{ji}\delta_k^t$$
$$+(g_{ik}-\eta_i\eta_k)f_j{}^t-(g_{jk}-\eta_j\eta_k)f_i{}^t)\nabla_t K. \tag{2.9}$$

Transvecting (2.9) with $f_l{}^j$ and using (2.6) and (2.7), we have

$$\nabla_k K_{ji}=-\eta_j(S_{ki}-(n-1)f_{ki})-\eta_i(S_{kj}-(n-1)f_{kj})$$
$$+\frac{1}{2(n+1)}((g_{jk}-\eta_j\eta_k)\delta_i^t+f_{ik}f_j{}^t+(g_{ik}-\eta_i\eta_k)\delta_j^t$$
$$+f_{jk}f_i{}^t+2(g_{ji}-\eta_j\eta_i)\delta_k^t)\nabla_t K. \tag{2.10}$$

3. Laplacian $\Delta(K_{ji}K^{ji})$ and $\Delta(Z_{ji}Z^{ji})$

We first define a tensor field Z_{ji} of the form

$$Z_{ji}=K_{ji}-\left(\frac{K}{n-1}-1\right)g_{ji}+\left(\frac{K}{n-1}-n\right)\eta_j\eta_i. \tag{3.1}$$

Then, taking account of (2.10), we have

$$(3.2)\quad \nabla_k Z_{ji} = -\eta_j\left(S_{ki} - \left(\frac{K}{n-1} - 1\right) f_{ki}\right) - \eta_i\left(S_{kj} - \left(\frac{K}{n-1} - 1\right) f_{kj}\right) - \frac{1}{n-1}(\nabla_k K) g_{ji} + \frac{1}{n-1}(\nabla_k K)\eta_j\eta_i + \frac{1}{2(n+1)}((g_{jk} - \eta_j\eta_k)\delta_i^t + f_{ik}f_j^t + (g_{ik} - \eta_i\eta_k)\delta_j^t + f_{jk}f_i^t + 2(g_{ji} - \eta_j\eta_i)\delta_k^t)\nabla_t K,$$

which and (2.8) give

$$(3.3)\quad \nabla_k\nabla_j Z_{ih} = -f_{ki}\left(S_{jh} - \left(\frac{K}{n-1} - 1\right) f_{jh}\right) - \eta_i\nabla_k\left(S_{jh} - \left(\frac{K}{n-1} - 1\right) f_{jh}\right) - f_{kh}\left(S_{ji} - \left(\frac{K}{n-1} - 1\right) f_{ji}\right) - \eta_h\nabla_k\left(S_{ji} - \left(\frac{K}{n-1} - 1\right) f_{ji}\right) - \frac{1}{n-1}(\nabla_k\nabla_j K) g_{ih} + \frac{1}{n-1}(\nabla_k\nabla_j K)\eta_i\eta_h + \frac{1}{n-1}(\nabla_j K)(f_{ki}\eta_h + \eta_i f_{kh}) - \frac{1}{2(n+1)}((f_{kj}\eta_i + \eta_j f_{ki})\delta_h^t - (\eta_h g_{kj} - \eta_j g_{kh}) f_i^t - f_{hj}(\eta_i\delta_k^t - \eta^t g_{ki}) + 2(f_{ki}\eta_h + \eta_i f_{kh})\delta_j^t + (f_{kh}\eta_j + \eta_h f_{kj})\delta_i^t - (\eta_i g_{kj} - \eta_j h_{ki}) f_h^t - f_{ij}(\eta_h\delta_k^t - \eta^t g_{hk}))\nabla_t K + \frac{1}{2(n+1)}((g_{ji} - \eta_j\eta_i)\delta_h^t + f_{hj}f_i^t + (g_{hj} - \eta_h\eta_j)\delta_i^t + f_{ij}f_h^t + 2(g_{ih} - \eta_i\eta_h)\delta_j^t)\nabla_k\nabla_t K.$$

Transvecting (3.3) with $g^{kj}Z^{ih}$ and making use of $Z_{ji}\eta^i = 0$ and $Z_i^i = Z_{ji}g^{ji} = 0$, we can easily see that

$$(3.4)\quad g^{kj}(\nabla_k\nabla_j Z_{ih}) Z^{ih} = -2 f_{si} S_h{}^s Z^{ih} + \frac{1}{n+1}(Z_k^t + f_{hk} f_i^t Z^{ih})\nabla^k\nabla_t K,$$

where $Z_j{}^i = Z_{jh}g^{hi}$ and $Z^{ji} = Z_h{}^i g^{hj}$.

On the other side, taking account of the skew-symmetry of S_{ji}, we obtain

$$f_{si}S_h{}^s Z^{ih} = K_{ih}Z^{ih} = K_{ih}K^{ih} - \left(\frac{K}{n-1} - 1\right)K + (n-1)\left(\frac{K}{n-1} - n\right)$$

and

$$f_{hk} f_i^t Z^{ih} = Z_k^t.$$

Substituting the last two equations in (3.4) implies

$$(3.5)\quad g^{kj}(\Delta_k\Delta_j Z_{ih}) Z^{ih} = -2K_{ih}K^{ih} + 2\left(\frac{K}{n-1} - 1\right)K - 2(n-1)\left(\frac{K}{n-1} - n\right) + \frac{2}{n+1}(\nabla_k(Z^{kt}\nabla_t K) - (\nabla_k Z_t^k)\nabla^t K).$$

Next, taking account of (3.1), we have by a straightforward computation

$$(\nabla_k Z_{ji})(\nabla^k Z^{ji}) = (\nabla_k K_{ji} - \frac{1}{n-1}(\nabla_k K) g_{ji} + \frac{1}{n-1}(\nabla_k K)\eta_j\eta_i + \left(\frac{K}{n-1}-n\right)(f_{kj}\eta_i + \eta_j f_{ki}))\ (\nabla^k K^{ji} - \frac{1}{n-1}(\nabla^k K) g^{ji} + \frac{1}{n-1}(\nabla^k K)\eta^j\eta^i + \left(\frac{K}{n-1}-n\right)(f^{kj}\eta^i + \eta^j f^{ki})),$$

which reduces to

$$(3.6)\qquad (\nabla_k Z_{ji})(\nabla^k Z^{ji}) = (\nabla_k K_{ji})(\nabla^k K^{ji}) - \frac{1}{n-1}(\nabla_k K)(\nabla^k K) + 4\left(\frac{K}{n-1}-n\right)(-K+n(n-1)) + 2(n-1)\left(\frac{K}{n-1}-n\right)^2$$

since $f_{kj}\eta_i \nabla^k K^{ji} = -K + n(n-1)$ which is a direct consequence of (2.2).

On the other hand we can also find by using (2.10)

$$(3.7)\qquad (\nabla_k K_{ji})(\nabla^k K^{ji}) = (\eta_j(S_{ki} - (n-1)f_{ki}) + \eta_i(S_{kj} - (n-1)f_{kj}) + \frac{1}{2(n+1)}(((-g_{jk} + \eta_j\eta_k)\delta_i^t - f_{ik} f_j^t + (-g_{ik} + \eta_i\eta_k)\delta_j^t - f_{jk} f_i^t + 2(-g_{ji} + \eta_j\eta_i)\delta_k^t)\nabla_t K)(\eta^j(S^{ki} - (n-1)f^{ki}) + \eta^i(S^{kj} - (n-1)f^{kj}) + \frac{1}{2(n+1)}((-g^{jk} + \eta^j\eta^k)g^{is} - f^{ik}f^{js} + (-g^{ji} + \eta^j\eta^i)g^{ks} - f^{jk}f^{is} + 2(-g^{ji} + \eta^j\eta^i)g^{ks})\nabla_s K)$$

$$= 2K_{ji}K^{ji} - 4(n-1)K + 2n(n-1)^2 + \frac{2}{n+1}(\nabla_t K)(\nabla^t K),$$

where we have used $S_{ji}S^{ji} = K_{ji}K^{ji} - (n-1)^2$, $f_{ji}S^{ji} = K - (n-1)$ and (2.6).

Finally we contract (3.2) with g^{kj}. Then by means of (2.6) we get

$$(3.8)\qquad (\nabla_k Z_i^k)\nabla^i K = \frac{n-3}{2(n-1)}(\nabla_t K)(\nabla^t K).$$

Hence the Laplacian of $Z_{ji}Z^{ji}$

$$(3.9)\qquad \frac{1}{2}\Delta(Z_{ji}Z^{ji}) = g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih} + (\nabla_k Z_{ji})(\nabla^k Z^{ji})$$

is given by

$$(3.10)\qquad \frac{1}{2}\Delta(Z_{ji}Z^{ji}) = \frac{2}{n+1}\nabla_k(Z^{kt}\nabla_t K)$$

with the aid of (3.5)–(3.8), and consequently we have

LEMMA 1 (see also [5]). *In a Sasakian manifold with constant scalar curvature whose C-Bochner curvature tensor vanishes identically,*

$$\Delta(Z_{ji}Z^{ji})=0.$$

Moreover, replacing the quantites about Z_{ji} by those of K_{ji} in (3.10), *we can easily find*

$$\frac{1}{2}\Delta(K_{ji}K^{ji})=\left(\frac{2}{n+1}K_{ji}+\frac{K-(n-1)}{n-1}g_{ji}\right)\nabla^j\nabla^i K+\frac{2}{n+1}\|\nabla_j K\|^2. \tag{3.11}$$

4. Main theorems

We first assume that M^n is a Sasakian manifold with vanishing C-Bochner curvature tensor whose scalar curvature is constant. Then, by Lemma 1, we have

$$\begin{aligned}\frac{1}{2}\Delta(Z_{ji}Z^{ji})&=g^{kj}(\nabla_k\nabla_j Z_{ih})Z^{ih}+(\nabla_k Z_{ji})(\nabla^k Z^{ji})\\&=g^{kj}\nabla_k(\nabla_i Z_{jh}+\eta_j(S_{ih}-\left(\frac{K}{n-1}-1\right)f_{ih})-\eta_i(S_{jh}-\left(\frac{K}{n-1}-1\right)f_{jh})\\&\quad-2\eta_h(S_{ji}-\left(\frac{K}{n-1}-1\right)f_{ji}))Z^{ih}+(\nabla_k Z_{ji})(\nabla^k Z^{ji})=0\end{aligned}$$

with the help of (3.2).

Applying Ricci's identity to the last equation and using $Z_i{}^i=0$ and $Z_{ji}\eta^i=0$, we can easily find

$$K_i{}^t Z_{th}Z^{ih}-K_{sih}{}^t Z_t{}^s Z^{ih}-3Z_{ih}Z^{ih}+(\nabla_k Z_{ji})(\nabla^k Z^{ji})=0 \tag{4.1}$$

with the aid of

$$\nabla^t Z_{th}=0,\qquad f_{si}S_h{}^s Z^{ih}=Z_{ih}Z^{ih}.$$

On the other hand, using (1.1) with $B_{kji}{}^h=0$, we can get

$$\begin{aligned}K_i{}^t Z_{th}Z^{ih}-K_{sih}{}^t Z_t{}^s Z^{ih}&=\frac{n-1}{n+3}Z_i{}^t Z_{th}Z^{ih}\\&\quad+\frac{1}{n+3}\left(\frac{n+3}{n+1}K+\frac{2(n-1)}{n+1}+2n+2\right)Z_{ji}Z^{ji}.\end{aligned} \tag{4.2}$$

Since

$$Z_{ji}Z^{ji}=K_{ji}K^{ji}-\frac{1}{n-1}K^2+2K-n(n-1), \tag{4.3}$$

from (3.6) and (3.7) we have

$$(\nabla_k Z_{ji})(\nabla^k Z^{ji})=2Z_{ji}Z^{ji}. \tag{4.4}$$

Hence, substituting (4.2) and (4.4) into (4.1), we have

$$\frac{n-1}{n+3}Z_i{}^t Z_{th}Z^{ih}+\frac{1}{n+1}(K+n-1)Z_{ji}Z^{ji}=0. \tag{4.5}$$

We introduce a lemma due to *M.* Okumura [3].

LEMMA F ([3]). *Let a_i, $i=1, 2, \cdots$, n be n real numbers satisfying*

$$\sum_{i=1}^{n} a_i=0 \quad \text{and} \quad \sum_{i=1}^{n} a_i^2=k^2$$

for a constant $k\geqq 0$. Then we have

$$-\frac{n-2}{\sqrt{n(n-1)}}k^3\leqq\sum_{i=1}^{n} a_i^3\leqq\frac{n-2}{\sqrt{n(n-1)}}k^3.$$

By Lemma F we have

LEMMA 2. *Put*

$$Z_{ji}=K_{ji}-\left(\frac{K}{n-1}-1\right)g_{ji}+\left(\frac{K}{n-1}-n\right)\eta_j\eta_i,$$
$$z^2=Z_{ji}Z^{ji},\quad Z^{ji}=Z_{st}g^{sj}g^{ti},\quad Z\geqq 0$$

and let a_i, $i=1, 2, \cdots$, n be eigenvalues of $Z_j{}^i=Z_{jh}g^{hi}$.

Then we have

$$-\frac{n-5}{\sqrt{2(n-1)(n-3)}}z^3\leqq\sum_{i=1}^{n} a_i^3\leqq\frac{n-5}{\sqrt{2(n-1)(n-3)}}z^3$$

Proof. Since $Z_j{}^i$ and $f_j{}^i$ are commutative, if $Z_t{}^iX^t=\alpha X^i$ and $X^i\perp\eta^i$, then $Z_t{}^if_s{}^tX^s=\alpha f_t{}^iX^t$. Therefore the matrix $(Z_j{}^i)$ is of the form

$$(Z_j{}^i)=\begin{pmatrix} a_1 & & & & & & & & \\ & a_2 & & & & & & & \\ & & \ddots & & & & & 0 & \\ & & & a_m & & & & & \\ & & & & a_1 & & & & \\ & 0 & & & & a_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & a_m & \\ & & & & & & & & 0 \end{pmatrix}$$

because of $Z_j{}^i\eta^j=0$, and consequently, since $Z_i{}^i=0$, we have

$$\sum_{i=1}^{m} a_i=0, \qquad 2\sum_{i=1}^{m} a_i^2=z^2,\quad n=2m+1.$$

Thus by means of Lemma F we can obtain

$$-\frac{n-5}{\sqrt{(n-1)(n-3)}}{}'z^3 \leqq \sum_{i=1}^{m} a_i^3 \leqq \frac{n-5}{\sqrt{(n-1)(n-3)}}{}'z^3, \tag{4.6}$$

where we put

$$\sum_{i=1}^{m} a_i^2 = {}'z^2.$$

On the other hand, $2'z^2=z^2$, or equivalently ${}'z^3=\frac{1}{2\sqrt{2}}z^3$, which together with $\sum_{i=1}^{n} a_i^3 = 2\sum_{i=1}^{m} a_i^3$ and (4.6) gives our lemma.

Finally, using Lemma 2, we prove

THEOREM. *Let M^n be a Sasakian manifold of dimension n (>3) with constant scalar curvature K whose C-Bochner curvature tensor vanishes. If the length of Ricci tensor is not greater that $K/\sqrt{n-2}$, then M^n is locally C-Fubinian.*

Proof. The equation (3.7) with $\nabla_t K=0$ gives

$$4(n-1)K \leqq 2K_{ji}K^{ji}+2n(n-1)^2,$$

that is,

$$2K \leqq \frac{1}{n-1}K_{ji}K^{ji}+n(n-1),$$

from which together with (4.3), we have

$$Z_{ji}Z^{ji} \leqq \frac{n}{n-1}K_{ji}K^{ji}-\frac{1}{n-1}K^2. \tag{4.7}$$

If the length of Ricci tensor is not greater than $K/\sqrt{n-2}$, it follows that

$$K_{ji}K^{ji} \leqq \frac{1}{n-2}K^2,$$

which and (4.7) imply

$$Z_{ji}Z^{ji} \leqq \frac{2}{(n-1)(n-2)}K^2,$$

or equivalently

$$z \leqq \frac{\sqrt{2}}{\sqrt{(n-1)(n-2)}}K. \tag{4.8}$$

Taking account of Lemma 2 and (4.8), we have from (4.5)

$$0=\frac{n-1}{n+3}Z_i{}^tZ_{th}Z^{ih}+\frac{1}{n+1}(K+n-1)Z_{ji}Z^{ji}$$

$$\geqq\left(\frac{K+n-1}{n+1}-\frac{n-1}{n+3}\frac{n-5}{\sqrt{2(n-1)(n-3)}}z\right)Z_{ji}Z^{ji}$$

$$\geqq\left(\left(\frac{1}{n+1}-\frac{n-5}{n+3}\frac{1}{\sqrt{(n-2)(n-3)}}\right)K+\frac{n-1}{n+1}\right)Z_{ji}Z^{ji},$$

from which, since $\left(\frac{1}{n+1}-\frac{n-5}{n+3}\frac{1}{\sqrt{(n-2)(n-3)}}\right)K\geqq 0,$

$$Z_{ji}=0$$

and consequently M^n is C-Einstein. Hence M^n is locally C-Fubinian.

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