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INFINITESIMAL PARALLEL AND NORMAL VARIATIONS OF HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

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0. Introduction

Chen [1], Yano [5, 6] have recently studied infinitesimal variations of submanifolds of Riemannian and Kaehlerian manifolds. Let $P^n(\mathbf{C})$ $(n\geqq 2)$ be a complex projective space of dimension n and C^{n+1} be the space of $(n+1)$-tuples of complex numbers $(z_1, \ldots, z_{n+1})$ and put

$$S^{2n+1}=\{(z_1, \ldots, z_{n+1})\in \mathbf{C}^{n+1} : \sum_{j=1}^{n+1}|z_j|^2=1\}.$$

For a positive number r we denote by $M_0'(2n, r)$ a hypersurface of an odd-dimensional sphere S^{2n+1} defined by

$$\sum_{j=1}^{n}|z_j|^2=r|z_{n+1}|^2, \quad \sum_{j=1}^{n+1}|z_j|^2=1.$$

Let π be the natural projection of S^{2n+1} onto $P^n(\mathbf{C})$. Then $M_0(2n-1, r)=\pi(M_0'(2n, r))$ is a connected complete real hypersurface of $P^n(\mathbf{C})$ with two constant principal curvatures. We call $M_0(2n-1, r)$ a geodesic hypersphere of $P^n(\mathbf{C})$, [4].

Recently, Takagi [4] has proved:

THEOREM A. *If M is a connected complete real hypersurface in a complex projective space $P^n(\mathbf{C})$ $(n\geq 2)$ with two constant principal curvatures, then M is a geodesic hypersphere.*

The main purpose of the present paper is to study hypersurfaces in a complex projective space concerning with the variations of the structure tensors induced on the hypersurface.

In §1, we state some preliminaries of an almost contact metric structure induced on a hypersurface of ambient Kaehlerian manifold.

In §2, we consider infinitesimal variations of a hypersurface of a complex projective space, and obtain some formulas concerning with variations of structure tensors of the almost contact metric structure induced on the hypersurface.

In §3, we consider a hypersurface in a complex projective space concerning with the variations of the structure tensors induced on the hypersurface, and prove the main theorem by using Theorem A.

1. Preliminaries

Let $M^{2m+2}(n>1)$ be a real $(2n+2)$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U : x^h\}$ and F_i^h be its almost complex structure tensor and g_{ji} its almost Hermitian metric tensor, where here and in the sequel the indices $h, i, j, \ldots$ run over the range $\{1', 2', \ldots, (2n+2)'\}$. Then we have

$$(1.1)\qquad F_i^t F_t^h = -\delta_i^h,\quad F_j^t F_i^s g_{ts} = g_{ji},$$

$$(1.2)\qquad \nabla_j F^h{}_i = 0,$$

where ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols Γ_{ji}^h formed with g_{ji}.

Let M^{2n+1} be a $(2n+1)$-dimensional orientable Riemannian manifold covered by a system of coordinate neighborhoods $\{V : y^a\}$ and g_{cb} be its fundamental metric tensor, where here and in the sequel the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, (2n+1)\}$. We assume that M^{2n+2} by the immersion $i : M^{2n+1} \to M^{2n+2}$ and identify $i(M^{2n+1})$ with M^{2n+1} itself. We represent the immersion by

$$(1.3)\qquad x^h = x^h(y^a)$$

and put

$$(1.4)\qquad B_b^h = \partial_b x^h,\quad (\partial_b = \partial/\partial y^b).$$

Then B_b^h are $2n+1$ linearly independent vectors of M^{2n+2} tangent to M^{2n+1}. Since the immersion i is isometric, we have

$$(1.5)\qquad g_{cb} = g_{ji} B_c^j B_b^i.$$

We represent the unit normal to M^{2n+1} by C^h, we then have

$$(1.6)\qquad g_{ji} B_b^j C^i = 0$$

and

$$(1.7)\qquad g_{ji} C^j C^i = 1.$$

Now the transform $F_i^h B_b^i$ of B_b^i by F_i^h can be written as

$$(1.8)\qquad F_i^h B_b^i = f_b^a B_a^h + f_b C^h,$$

where f_b^a is a tensor field of type of $(1, 1)$ and f_b a 1-form of M^{2n+1} and the transform $F_i^h C^i$ of C^i by F_i^h, being orthogonal to C^h, can be written as

(1.9) $$F_i{}^hC^i=-f^aB_a{}^h,$$

where $f^a=f_bg^{ba}$ is a vector field of M^{2n+1}, g^{ba} being contravariant components of g_{cb}.

Applying F to the both sides of (1.8), (1.9), respectively, and using (1.1), (1.2), (1.5), (1.6), (1.7), (1.8) and (1.9), we have

(1.10) $$\begin{cases} f_b{}^ef_e{}^a=-\delta_b^a+f_bf^a, \quad f_b{}^ef_e=0, \\ f_e{}^af^e=0, \quad f_ef^e=1, \\ f_c{}^ef_b{}^dg_{ed}=g_{cb}-f_cf_b. \end{cases}$$

Equations (1.10) show that the set $(f_b{}^a, g_{cb}, f_b)$ defines the so-called almost contact metric structure on M^{2n+1}. We denote by Γ_{cb}^a the Christoffel symbols formed with g_{cb}. Then it is well known that Γ_{ji}^h and Γ_{cb}^a are related by

(1.11) $$\Gamma_{cb}^a=(\partial_cB_b{}^h+\Gamma_{ji}^hB_{cb}{}^{ji})B_h{}^a,$$

where $B_{cb}{}^{ji}=B_c{}^jB_b{}^i$ and $B_h{}^a=B_b{}^ig_{ih}g^{ba}$.

We define the van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ along M^{2n+1} by

(1.12) $$\nabla_cB_b{}^h=\partial_cB_b{}^h+\Gamma_{ji}^hB_{cb}{}^{ji}-\Gamma_{cb}^aB_a{}^h$$

and that of C^h by

(1.13) $$\nabla_cC^h=\partial_cC^h+\Gamma_{ji}^hB_c{}^jC^i.$$

Then equations of Gauss can be written as

(1.14) $$\nabla_cB_b{}^h=h_{cb}C^h,$$

where h_{cb} is the second fundamental tensor of M^{2n+1} and those of Weingarten as

(1.15) $$\nabla_cC^h=-h_c{}^aB_a{}^h,$$

where $h_c{}^a=h_{cb}g^{ba}$.

Differentiating (1.8) and (1.9) covariantly along M^{2n+1} and taking account of (1.2), (1.14) and (1.15), we have

(1.16) $$\nabla_cf_b{}^a=-h_{cb}f^a+h_c{}^af_b, \quad \nabla_cf_b=-h_{ce}f_b{}^e.$$

If we put $f_{cb}=f_c{}^eg_{eb}$, then we easily see that

(1.17) $$f_{cb}=-f_{bc}.$$

Equations of the Gauss and Codazzi are respectively given by

(1.18) $$K_{dcb}{}^a=K_{kji}{}^hB^{kjia}{}_{dcbh}+h_d{}^ah_{cb}-h_c{}^ah_{db}$$

and

(1. 19) $$K_{kji}{}^{h}B^{kji}{}_{dcb}C_h=\nabla_d h_{cb}-\nabla_c h_{db},$$

$K_{dcb}{}^{a}$ and $K_{kji}{}^{h}$ being curvature tensors of M^{2n+1} and M^{2n+2} respectively, where $B^{kjia}{}_{dcbh}=B_d{}^kB_c{}^jB_b{}^iB_h{}^a$, $B^{kji}{}_{dcb}=B_d{}^kB_c{}^jB_b{}^i$ and $C_h=C^ig_{ih}$.

Now we suppose that the ambient Kaehlerian manifold M^{2n+2} is a complex projective space. Then its curvature tensor has components of the form

(1. 20) $$K_{kji}{}^{h}=\delta_k^h g_{ji}-\delta_j^h g_{ki}+F_k{}^hK_{ji}-F_j{}^hF_{ki}-2F_{kj}F_i{}^h,$$

where $F_{ji}=F_j{}^tg_{ti}$ [2].

Substituting (1. 20) into (1. 17) and (1. 18), equations of the Gauss and Codazzi reduces respectively to

(1. 21) $$K_{dcb}{}^{a}=\delta_d^a g_{cb}-\delta_c^a g_{db}+f_d{}^af_{cb}-f_{db}f_c{}^a-2f_{dc}f_b{}^a+h_d{}^ah_{cb}-h_{db}h_c{}^a,$$

and

(1. 22) $$\nabla_d h_{cb}-\nabla_c h_{db}=f_df_{cb}-f_cf_{db}-2f_bf_{dc}.$$

If the almost contact metric structure $(f_b{}^a, g_{cb}, f_b)$ satisfies

(1. 23) $$\nabla_c f_b-\nabla_b f_c=2\alpha f_{cb}\quad (\alpha=\text{non-zero function}),$$

then the structure is said to be *α-contact.*

Differentiating (1. 23) covariantly, and using (1. 16) and the Bianchi identity, we have

(1. 24) $$(\nabla_d\alpha)f_{cb}+(\nabla_c\alpha)f_{bd}+(\nabla_b\alpha)f_{dc}=0.$$

Thus we can easily see that

(1. 25) $$\alpha=\text{constant}$$

because of (1. 10).

Substituting (1. 16) into (1. 23), we have

(1. 26) $$f_c{}^eh_e{}^a+h_c{}^ef_e{}^a=2\alpha f_c{}^a.$$

Transvecting (1. 26) with $f^cf_a{}^b$ and using (1. 10), we find

(1. 27) $$h_{ce}f^e=hf_c,$$

where we have put $h=h_{cb}f^cf^b$.

Differentiating (1. 27) covariantly, we get

(1. 28) $$(\nabla_b h_{ce})f^e+h_{ce}\nabla_b f^e=(\nabla_b h)f_c+h\nabla_b f_c,$$

from which, taking the skew-symmetric part and using (1. 22),

(1. 29) $$h_{ce}\nabla_b f^e=\frac{1}{2}h(\nabla_b f_c-\nabla_c f_b)+f_{bc}+\frac{1}{2}\{(\nabla_b h)f_c-(\nabla_c h)f_b\}.$$

Substituting (1. 29) into (1. 28), we obtain

$$(1.30)\quad (\nabla_b h_{ce}) f^e = \frac{1}{2} h(\nabla_b f_c + \nabla_c f_b) - f_{bc} + \frac{1}{2}\{(\nabla_c h) f_b + (\nabla_b h) f_c\}.$$

If we substitute (1. 30) into (1. 28) and take account of (1. 23), we find

$$(1.31)\quad (\nabla_b h) f_c - (\nabla_c h) f_b + 2\alpha h f_{bc} = -2 f_{bc} - 2 h_c{}^e h_{ba} f_e{}^a,$$

from which, transvecting f^c,

$$(1.32)\quad \nabla_c h = \gamma f_c,$$

where $\gamma = (\nabla_b h) f^b$, which shows that

$$(1.33)\quad (\nabla_b \gamma) f_c - (\nabla_c \gamma) f_b + 2\alpha\gamma f_{bc} = 0.$$

Applying f^{bc} to (1. 33) and using (1. 10), we have $2\alpha\gamma f_{bc} f^{bc} = 0$. Since f_{bc} has the maximal rank and α is non-zero function, we find $\gamma = 0$. Thus it follows from (1. 32) that

$$(1.34)\quad h = \text{constant}.$$

2. Infinitesimal variations of hypersurfaces of a complex projective space

We now consider an infinitesimal variation of the hypersurface M^{2n+1} in M^{2n+2} given by

$$(2.1)\quad \bar{x}^h = x^h + \xi^h(y)\varepsilon,$$

ξ^h being a vector field of M^{2n+2} defined along M^{2n+1}, where ε is an infinitesimal. We then have

$$(2.2)\quad \bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon,$$

where $\bar{B}_b{}^h (= \partial_b \bar{x}^h)$ are $2n+1$ linearly independent vectors tangent to the varied hypersurface at the varied point $(\bar{x}^h)$.

We displace vectors $\bar{B}_b{}^h$ parallelly form the varied point $(\bar{x}^h)$ to the original point (x^h) and obtain

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}{}^h (x + \xi\varepsilon) \xi^j \bar{B}_b{}^i \varepsilon,$$

or

$$(2.3)\quad \tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon,$$

neglecting the terms of order higher than one with respect to ε, where

$$(2.4)\quad (\nabla_b \xi^h) = \partial_b \xi^h + \Gamma_{ji}{}^h B_b{}^j \xi^i.$$

In the sequel we always neglect terms of order higher than one with respect to ε.

Putting

$$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h, \tag{2.5}$$

we have

$$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon. \tag{2.6}$$

If we put

$$\xi^h = \xi^a B_a{}^h + \lambda C^h, \tag{2.7}$$

ξ^a and λ being respectively a vector field and a scalar function on M^{2n+1}, we have

$$\nabla_b \xi^h = (\nabla_b \xi^a - \lambda h_b{}^a) B_a{}^h + (\nabla_b \lambda + h_{be} \xi^e) C^h. \tag{2.8}$$

We denote by $\bar{C}^h$ the unit normal to the varied hypersurface. We displace $\bar{C}^h$ parallelly from the point $(\bar{x}^h)$ to (x^h) and obtain

$$\tilde{C}^h = \bar{C}^h + \Gamma_{ji}{}^h (x + \xi\varepsilon) \xi^j \bar{C}^i \varepsilon. \tag{2.9}$$

We put

$$\delta C^h = \tilde{C}^h - C^h. \tag{2.10}$$

Then δC^h, being orthogonal to C^h, is of the form

$$\delta C^h = \eta^a B_a{}^h, \tag{2.11}$$

η^a being a vector field on M^{2n+1}. Thus from (2.9), (2.10) and (2.11) we have

$$\bar{C}^h = C^h - \Gamma_{ji}{}^h \xi^j C^i \varepsilon + \eta^a B_a{}^h \varepsilon. \tag{2.12}$$

Now applying the operator δ to $B_b{}^j C^i g_{ji} = 0$ and using (2.6), (2.11) and $\delta g_{ji} = 0$, we find

$$(\nabla_b \xi^j) C^i g_{ji} + B_b{}^i \eta^a B_a{}^i g_{ji} = 0,$$

from which, using (2.8),

$$\eta_b = -(\nabla_b \lambda + h_{ba} \xi^a), \tag{2.13}$$

where $\eta_b = \eta^c g_{cb}$.

Thus (2.11) can be written as

$$\delta C^h = -(\nabla^a \lambda + h_b{}^a \xi^b) B_a{}^h \varepsilon, \tag{2.14}$$

where $\nabla^a = g^{ab} \nabla_b$ and (2.12) as

$$\bar{C}^h = C^h - \Gamma_{ji}{}^h \xi^j C^i \varepsilon - (\nabla^a \lambda + h_b{}^a \xi^b) B_a{}^h \varepsilon. \tag{2.15}$$

From (1.5), (1.8), (1.9) and (1.16) we can prove that [4]

$$
(2.16)\quad \begin{cases} \delta f_b{}^a=[Lf_b{}^a+\lambda(f_b{}^e h_e{}^a-h_b{}^e f_e{}^a)+f_b(\nabla^a\lambda)-(\nabla_b\lambda)f^a]\varepsilon, \\ \delta f_b=[Lf_b-\lambda h_b{}^e f_e-f_b{}^e\nabla_e\lambda]\varepsilon, \\ \delta f^a=[Lf^a+(\nabla^e\lambda)f_e{}^a+\lambda f^e h_e{}^a]\varepsilon, \\ \delta g_{cb}=[Lg_{cb}-2\lambda h_{cb}]\varepsilon, \end{cases}
$$

L being the operator of Lie derivation with respect to ξ^a.

When the tangent space at a point (x^h) of the original hypersurface and that the corresponding point $(\bar{x}^h)$ of the varied hypersurface are parallel, we say that the variation is *parallel*. If $\xi^a=0$, that is, if the variation vector ξ^h is normal to the hypersurface, then we say that the variation is *normal* [5]. Thus, under the acting of an infinitesimal parallel and normal variation $\bar{x}^h=x^h+\lambda C^h\varepsilon$ (λ=constant), (2.16) becomes

$$
(2.17)\quad \begin{cases} \delta f_b{}^a=\lambda(f_b{}^e h_e{}^a-h_b{}^e f_e{}^a)\varepsilon, \\ \delta f_b=-\lambda h f_b\varepsilon, \\ \delta f^a=\lambda h f^a\varepsilon, \\ \delta g_{cb}=-2\lambda h_{cb}\varepsilon. \end{cases}
$$

On the other hand, transvecting (1.23) with $f_a{}^c$ and using (1.10) and (1.23), we have

$$
(2.18)\qquad h_{cb}+h_{ed}f_c{}^e f_b{}^d=2\alpha g_{cb}-(h-2\alpha)f_c f_b.
$$

Transvecting this with g^{cb} and making use of (1.10), we obtain

$$
(2.19)\qquad h_e{}^e=2n\,\alpha+h,
$$

where $h_e{}^e=h_{ed}g^{ed}$.

From the last equation of (2.17), we have

$$
(2.20)\qquad \delta g^{cb}=(2\lambda\, h^{cb})\varepsilon.
$$

Moreover, it is well known that the variaton of the second fundamental tensor is given by [5]

$$
(2.21)\qquad \delta h_{cb}=[Lh_{cb}+\nabla_c\nabla_b\lambda+\lambda(K_{kji}{}^h C^k B^{ji}{}_{cb}C_h-h_{ce}h_b{}^e)]\varepsilon.
$$

If the ambient manifold is a complex projective space, then (2.21) reduces to

$$
(2.22)\qquad \delta h_{cb}=\lambda(g_{cb}+3f_c f_b-h_{ce}h_b{}^e)\varepsilon
$$

with the aid of (1. 8), (1. 9) and (1. 20).

Substituting (2. 20) into (2. 17) and using (1. 26), we find

$$\delta f_{cb} = -(2\alpha f_{cb})\varepsilon. \tag{2.23}$$

Applying the operator δ to (1. 27) and (2. 19) respectively, and taking account of (2. 17), (2. 20) and (2. 22), we have

$$\begin{cases} \delta h = \lambda(h^2+4)\varepsilon, \\ \delta h_e{}^e = \lambda(h_{ed}h^{ed}+2n+4)\varepsilon. \end{cases} \tag{2.24}$$

If we apply the operator δ to (2. 19) and use (2. 24), we obtain

$$\delta\alpha = \frac{\lambda}{2n}(h_{cb}h^{cb}-h^2+2n)\varepsilon. \tag{2.25}$$

3. Infinitesimal variations of α-contact hypersurfaces of a complex projective space

We consider parallel and normal variations of hypersurfaces of a complex projective space.

Now, applying the operator δ to (1. 26) and substituing (2. 17), (2. 22) and (2. 25), we have

$$2h_c{}^d h_{be} f_d{}^e + \frac{1}{n}(h_{cb}h^{cb}-h^2-4n\alpha^2)f_{cb} = 0, \tag{3.1}$$

from which, transvecting $f_a{}^b$ and using (2. 18),

$$h_c{}^e h_{be} - 2\alpha h_{cb} - A g_{cb} + (A-h^2+2\alpha h)f_c f_b = 0, \tag{3.2}$$

where we have put $A = \frac{1}{2n}(h_{cb}h^{cb}-h^2-4n\alpha^2)$.

Differentiating (3. 2) covariantly, we have

$$\begin{aligned}(\nabla_d h_c{}^e)h_{be} + h_c{}^e \nabla_d h_{be} &= 2\alpha\nabla_d h_{cb} + (\nabla_d A)g_{cb} - (\nabla_d A)f_c f_b \\ &\quad -(A-h^2+2\alpha h)\ \{(\nabla_d f_c)f_b + f_c(\nabla_d f_b)\},\end{aligned}$$

from which, transvecting with f^b and using (1. 16) and (1. 30),

$$\begin{aligned}(2A+2\alpha h-h^2)\nabla_d f_c + (h^2+2)\nabla_c f_d + (4\alpha-2h-Ah)f_{dc} \\ + hh_{da}h_{ce}f^{ae} = 0,\end{aligned} \tag{3.3}$$

or, taking symmetric parts,

$$(A+\alpha h+1)(\nabla_b f_c + \nabla_c f_b) = 0. \tag{3.4}$$

If we suppose that $A+\alpha h+1$ does not vanish, then we have

$$\nabla_b f_c + \nabla_c f_b = 0, \tag{3.5}$$

Substituting (3. 5) into (1. 26) and using (1. 26), we have

(3. 6) $$h_{ce}f_a{}^e=\alpha f_{ac}.$$

Transvecting (3. 6) with $f_b{}^a$ and taking account of (1. 10), we find

(3. 7) $$h_{cb}=\alpha g_{cb}.$$

Substituting (3. 7) into (1. 22), we have

$$f_d f_{cb}-f_c f_{db}-2f_b f_{dc}=0.$$

It contradicts the fact that f_{cb} has the maximal rank. Therefore, it follows that $A+\alpha h+1=0$. Consequently (3. 2) reduces to

$$h_c{}^e h_e{}^a-2\alpha h_c{}^a+(\alpha h+1)\delta_c{}^a-(h^2-\alpha h+1)f_c f^a=0.$$

Thus, the second fundamental tensor $h_c{}^a$ has three constant eigenvalues h, $\alpha\pm\sqrt{\alpha^2-\alpha h-1}$ because α and h are both constant. But, since the eigenvalues of $h_c{}^a$ are real and the discriminant of $\alpha^2-\alpha h-1$ is positive, we find $\alpha^2-\alpha h-1=0$. Hence the hypersurface has two distinct constant principal curvatures α and h.

According to Theorem A, we have

THEOREM. *If a complete hypersurface of a complex projective space admits an infinitesimal parallel and normal variation which preserves*

$$f_c{}^e h_e{}^a+h_c{}^e f_e{}^a=2\alpha f_c{}^a,$$

then it is a geodesic hypersphere.

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