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POWER INVARIANCE OF $R[[X_1, \cdots, X_n]]$

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1. Introduction

O'Malley [8] raised the following question: *If* R *and* S *are rings such that* $R[[X]] \cong S[[X]]$, *must* $R \cong S$? Hamann [4] showed an example of two nonisomorphic rings R and S whose formal power series rings $R[[X]]$ and $S[[X]]$ are isomorphic. A ring R is called *power invariant* if whenever S is a ring such that the formal power series rings $R[[X]]$ and $S[[X]]$ in an indeterminate X over R and S are isomorphic, then R and S are isomorphic. Several authors [4, 6, 7, 8] imposed some condition on a ring R so that R should be power invariant.

A commutative ring R with 1 is power invariant if $J(R)$, the Jacobson radical of R, is nilpotent [6], and Hamann [4] proved that R is power invariant if $J(R)$ is nil. Recently this author [7] showed that if $J(R)$ is nil, $R[[X]]$ is power invariant. Let $R^{((n))} = R[[X_1, \ldots, X_n]]$ be the formal power series ring in n indeterminates $X_1, \ldots, X_n$ over a ring R. Then naturally the following question arises: *If* $J(R)$ *is nil, is* $R^{((n))}$ *power invarant for* $n \geq 2$? Andy Magid's counterexample [4] forces the answer to be negative.

In this paper we investigate power invariance of $R^{((n))}$ for the case $n \geq 2$. If $\alpha_1, \cdots, \alpha_n \in R^{((n))}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ will denote the ideal of $R^{((n))}$ generated by $\alpha_1, \cdots, \alpha_n$. Let $(R^{((n))}, \alpha)$ denote the topological ring $R^{((n))}$ with the α-adic topology. $(R^{((n))}, \alpha)$ is Hausdorff if and only if $\bigcap_{n=1}^{\infty} \alpha^n = (0)$. If this is the case, $R^{((n))}$ is metrizable. and $(R^{((n))}, \alpha)$ is said to be *complete* if each Cauchy sequence of $R^{((n))}$ converges in $R^{((n))}$.

Throughout this paper all rings are assumed to be commutative and to have identity elements, and our terminology and notation will be that of [2] and [7].

2. Preliminaries

Let $\beta \in R^{((n))}$, then β is uniquely expressible in the from $\sum_{j=0}^{\infty} \beta_i$, where

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$\beta_j \in R[X_1, \ldots, X_n]$ for each j such that β_j is 0 or a homogeneous polynomial of degree j in $X_1, \cdots, X_n$ over R. We call $\sum_{j=0}^{\infty} \beta_j$ the homogeneous decomposition of β, and β_j the j-th homogeneous component of β for each j. The following theorem (Lemma 3.1 and Theorem 5.6 in [2]) will be needed for our main results.

THEOREM 1. *Let $\alpha_i = \sum_{j=0}^{\infty} \alpha^{(i)} \in R^{((n))}$, $i=1, \cdots, n$, be homogeneous decompositions of elements of $R^{((n))}$. There exists an R-automorphism ϕ of $R^{((n))}$ such that $\phi(X_i) = \alpha_i$ for each i if and only if the following conditions are satisfied:*

(1) *$(R^{((n))}, \alpha)$ is a complete Hausdorff space;*

(2) *$R\alpha_1^{(1)} + \cdots + R\alpha_1^{(n)} = RX_1 + \cdots + RX_n$. (or the $n \times n$ matrix $(a_{1j}^{(i)})_{ij}$ is a unit of $M_n(R)$, where $\alpha_1^{(i)} = a_{11}^{(i)}X_1 + a_{12}^{(i)}X_2 + \cdots + a_{1n}^{(i)}X_n$, $a_{1j}^{(i)} \in R$ for each i and j.)*

Let I_c(R) denote the set of all $a \in R$ such that there is an R-homomorphism $\sigma: R[[X]] \rightarrow R$ with $\sigma(X) = a$. Then $I_c(R)$ is an ideal of R which is contained in $J(R)$ and contains the nil-radical of R. (By Theorem E, [1]). $I_c(R)$ may be properly contained in $J(R)$ and it may properly contain the nilradical of R. See [1] for such examples.

It was proved in [3] that $I_c(R^{((n))}) = I_c(R) + (X_1, \ldots, X_n)$ for any ring R; therefore for any positive integer n, $I_c(R^{((n))})$ can not be nil. Note that $I_c(R)$ is the set of all elements a in R such that there exists an R-automorphism of $R[[X]]$ taking X to $X+\alpha$. (By Theorem D, [1]).

3. Main Results

LEMMA 2. *Let A and B be ideals in $R[[X_1, \ldots, X_n]]$ with B contained in the Jacobson radical of $R[[X_1, \ldots, X_n]]$. Then if $X_k \in AB$ for some k, $1 \leq k \leq n$,* then $1 \in A$.

Proof: Suppose $X_k \in AB$ and let $X_k = \sum_{i=1}^{n} f_i g_i$ where $f_i \in A$ and $g_i \in B$ for each $i=1, \ldots, n$. Let $f_i = \sum_{j=0}^{\infty} f^{(i)}$ and $g_i = \sum_{j=0}^{\infty} g_j^{(i)}$ be homogeneous decompositions of f_i and g_i, respectively. Let $f_1^{(i)} = f_{11}^{(i)}X_1 + \cdots + f_{1n}^{(i)}X_n$ and $g_1^{(i)} = g_{11}^{(i)}X_1 + \cdots + g_{1n}^{(i)}X_n$ where $f_{1j}^{(i)}$ and $g_{1j}^{(i)}$ are elements of R for each $j=1, \cdots, n$. Taking the X_k-coefficient of both sides of $X_k = \sum_{i=1}^{n} f_i g_i$ we get $1 = \sum_{i=1}^{n} f_{1k}^{(i)} g_0^{(i)} + \sum_{i=1}^{n} f_0^{(i)} g_{1k}^{(i)}$. Since $g_0^{(i)} \in J(R)$ for each $i=1, \ldots, n$, $\sum_{i=1}^{n} f_0^{(i)} g_{1k}^{(i)}$ is a unit of R; therefore, the ideal $(f_0^{(1)}, \ldots, f_0^{(n)})$ contains 1 and hence the ideal $(f_1, \ldots, f_n)$ contains 1. Thus $1 \in$ A.

LEMMA 3. *If η is a nilpotent element of $R[[X]]$, then there exists an R-automorphism ϕ of $R[[X[]$ such that $\phi(X) = X + \eta$.*

Proof: Suppose that η is a nilpotent element of $R[[X]]$ and let $\eta=\sum_{i=0}^{\infty} a_iX^i$. Then a_i is nilpotent for each $i=1, 2, \ldots$. Then $X+\eta=a_0+(1+a_1)X+\sum_{i=2}^{\infty}a_iX^i$. Since a_0 is nilpotent and $1+a_1$ is a unit of R, by Theorem 1 there is an R-automorphism ϕ of $R[[X]]$ such that $\phi(X)=X+\eta$.

THEOREM 4. *Let R be a ring such that $I_c(R)$ is nil. Then $R[[X_1, \ldots, X_{n-1}]]$ is power invariant if any unimodular row vector $[u_1, \cdots, u_n]$, $u_i \in R$ for $i=1, \ldots, n$, can be complected to an $n\times n$ matrix of determinant one.*

Proof: Suppose that $I_c(R)$ is nil and unimodular row vector $[u_1, \ldots, u_n]$ can be completed to an $n\times n$ matrix whose determinant is 1. To prove the theorem it suffices to show that $R[[X_1, \ldots, X_{n-1}, X_n]] = S[[Y]]$ implies $R[[X_1, \ldots, X_{n-1}]]\cong S$. Here X_n and Y are indeterminates over $R[[X_1, \cdots, X_{n-1}]]$ and a ring S, respectively.

Let $W=R[[X_1, \ldots, X_n]] = S[[Y]]$ and $Y=a+\sum_{i=1}^{n}U_iX_i$ where $a\in R$ and $U_i \in W$ for each $i=1, \cdots, n$. Since $I_c(W)=I_c(R)+\sum_{i=1}^{n}X_iW=I_c(S)+YW$, we note $a \in I_c(R)$. Thus a is a nilpotent element of R and so it is a nilpotent element of $S[[Y]]$. By Lemma 3, there is an S-automorphism of $S[[Y]]$ taking Y to $Y-a$. We may therefore assume $a=0$. We then have $Y \in (U_1, \ldots, U_n)(X_1, \ldots, X_n)$. Clearly, $(X_1, \ldots, X_n)$ is contained in $J(W)$, so it is contained in $J(S[[Y]])$. Then by Lemma 2, the ideal $(U_1, \ldots, U_n)$ contains 1.

Let $u_1, \ldots, u_n$ be the constant terms of $U_1, \ldots, U_n$ considered as elements of $R[[X_1, \ldots, X_n]]$. Then the ideal $(u_1, \ldots, u_n)$ contains 1, i. e., $[u_1, \ldots, u_n]$ is a unimodular vector, so it can be completed to an $n\times n$ matrix (u_{ij}) of determinant 1, where $u_{nj}=u_j$ for each $j=1, \ldots, n$. Then by Theorem 1, there exists an R-automorphism of $R^{((n))}$ taking X_n to Y. Then

$$R[[X_1, \ldots, X_{n-1}]] \cong W/(X_n) \cong W/(Y) = S[[Y]]/(Y) \cong S.$$

Thus $R[[X_1, \ldots, X_{n-1}]]$ is power invariant.

In case $n=1$ and $n=2$ we see that any unimodular vectors $[u]$ and $[u_1, u_2]$ can be completed to 1×1 matrix and 2×2 matrix of determinant one, respectively. Thus we have the following corollary which appeared as Theorem 3 and 5 in [7],

COROLLARY 5. *If R is a ring such that $I_c(R)$ is nil, then R and $R[[X]]$ are power invariant.*

On the other hand for $n\geq 3$ the corollary will not hold. In other words $R[[X_1, \ldots, X_{n-1}]]$, $n\geq 3$, may not be power invariant even if $I_c(R)$ is nil. Let R be the reals and let X, Y, Z be independent indeterminates over R.

Let $A=R[X, Y, Z]/(X^2+Y^2+Z^2-1)=R[x, y, z]$. Then A is a Noetherian ring with zero Jacobson radical which has a finitely generated nonfree module P such that $P\oplus A \cong A^3$. Taking symmetric algebras of both sides yields

$S_A(P)[T] \cong A[X_1, X_2, X_3]$, where $S_A(P)$ is the symmetric algebra of P and T, $\{X_1, X_2, X_3\}$ are indeterminates over $S_A(P)$ and A, respectively, but $S_A(P)$ is not isomorphic to $A[X_1, X_2]$. This is Melvin Hochster's counter-example to the question whether $R[X] \cong S[X]$ implies $R \cong S$, where X is an indeterminate over rings R and X. See [5] for more detail.

Let $\hat{S}_A(P)$ be the completion of $S_A(P)$ with respect to the ideal generated by P. Then the completion of $S_A(P)[T]$ with respect to the ideal (P, T) is $\hat{S}_A(P)[[T]]$ and the completion of $A[X_1, X_2, X_3]$ with respect to the ideal (X_1, X_2, X_3) is $A[[X_1, X_2, X_3]]$ and clearly $\hat{S}_A(P)[[T]] \cong A[[X_1, X_2, X_3]]$. We claim that $\hat{S}_A(P)$ is not isomorphic to $A[[X_1, X_2]]$.

For suppose $\hat{S}_A(P) \cong A[[X_1, X_2]]$. Since $S_A(P)$ is a Noetherian ring, $\hat{S}_A(P)/P\hat{S}_A(P) \cong S_A(P)/PS_A(P) \cong A$ whose Jacobson radical is zero. Clearly $P\hat{S}_A(P)$ is contained in $J(\hat{S}_A(P))$; therefore, $J(\hat{S}_A(P)) = P\hat{S}_A(P)$. The associated graded rings of $\hat{S}_A(P)$ and $A[[X_1, X_2]]$ with respect to their Jacobson radicals have to be isomorphic but one of these is $A \oplus P\hat{S}_A(P)/P^2\hat{S}_A(P) \oplus \cdots$ which is isomorphic to $A \oplus PS_A(P)/P^2S_A(P) \oplus \cdots$ which is isomorhpic to $S_A(P)$, and the other is $A \oplus (X_1, X_2)/(X_1, X_2)^2 \oplus \cdots$ which is isomorphic to $A[X_1, X_2]$. Thus $S_A(P) \cong A[X_1, X_2]$ which is impossible, hence $\hat{S}_A(P)$ is not isomorphic to $A[[X_1, X_2]]$. So $A[[X_1, X_2]]$ is not power invariant, although $J(A) = I_c(A) = (0)$. This is the Andy Magid's counter example [4] to the question whether $R[[X]] \cong S[[X]]$ implies $R \cong S$.

This example indicates that $R[[X_1, \cdots, X_n]]$, $n \geq 2$, may not be power invariant even if $I_c(R)$ is nil.

References

1. P. Eakin and A. Sathaye, *R-endomorphisms of* $R[[X]]$ *are essentially continuous*, Pacific J. Math., **66**(1976), 83–87.
2. R. Gilmer and M. O'Malley, *R-endomorphisms of* $R[[X_1, X_n]]$, J. Algebra, **48** (1977), 30–45.
3. R. Gilmer and M. O'Malley, *On R-homomorphisms of power series rings*, Advances in Math. (to appear).
4. E. Hamann, *On power-invariance*, Pacific J. Math. **61** (1975), 153–159.
5. M. Hochster, *Nonuniqueness of coefficient rings in a polynomial ring*, Proc. Amer. Math. Soc., **34** (1972), 81–82.
6. J. H. Kim, *Power invariant ring*, Pacific J. Math., **51**. (1974), 207–213.
7. J. H. Kim, *On isomorphisms of power series rings*, Arch. Math. (to appear).
8. M. O'Malley, *Isomorphic power series rings*, Pacific J. Math., **41** (1972), 503–512.

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