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ON A DECOMPOSITION OF CONTRAVARIANT C^*-ANALYTIC VECTOR FIELDS IN A COMPACT COSYMPLECTIC η-EINSTEIN MANIFOLD

BY SANG-SEUP EUM AND UN KYU KIM

0. Introduction

Y. Matsushima proved the following theorem (Yano[4]).

THEOREM A. *In a compact Kaehler-Einstein space* ($K>0$), *any contravariant analytic vector* v^h *is uniquely decomposed into the form* $v^h=p^h+F_i{}^hq^i$, *where* p^h *and* q^h *are both Killing vectors.*

As an analogue of this theorem A in a Sasakian manifold, I. Sato proved the following theorem (Sato[3]).

THEOREM B. *In a compact C-Einstein space such that* $R_{ji}=2(ag_{ji}+b\eta_j\eta_i)$ *and* $a+2\neq 0$, *a special C-analytic vector field* u^h *can be decomposed in the form* $u^h=v^h+\varphi_t{}^hw^h$, *where* v^h *is a C-Killing and* w^h *is a special C-Killing.*

The main purpose of the present paper is to find a cosymplectic analogue of the theorem A.

In §1, we state some of fundamental formulas in cosymplectic manifolds to fix our notations and in §2, we study contravariant C^*-analytic vector fields in a cosymplectic manifold which corresponds to the contravariant C-analytic vector fields in a Sasakian manifold (Sato [3]). In §3, we study a unique decomposition of contravariant C^*-analytic vector fields in a compact cosymplectic η-Einstein manifold.

1. Cosymplectic manifolds

Let M be a $(2n+1)$-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U;x^h\}$ in which there are given a tensor field $\varphi_i{}^h$ of type $(1,1)$, a vector field ξ^h and a 1-form η_h satisfying

(1.1) $$\varphi_j{}^i\varphi_i{}^h=-\delta_j^h+\eta_j\xi^h,\quad \varphi_j{}^h\xi^j=0,\quad \eta_i\varphi_j{}^i=0,\quad \eta_i\xi^i=1,$$

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where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$. Such a set of a tensor field φ of type (1, 1), a vector field ξ and a 1-form η is called an *almost contact structure* and a manifold with an almost contact structure an *almost contact manifold.*

If, in an almost contact manifold, there is given a Riemannian metric g_{ji} such that

(1.2) $$g_{ts}\varphi_j{}^t\varphi_i{}^s = g_{ji} - \eta_j\eta_i, \quad \eta_i = g_{ih}\xi^h,$$

then the almost contact structure is said to be *metric* and the manifold is called an *almost contact metric manifold.*

Comparing the first equations of (1.1) and (1.2), we see that $\varphi_{ji} = \varphi_j{}^t g_{ti}$ is skew-symmetric.

Since, in an almost contact metric manifold, we have the second equation of (1.2), we shall write η^h instead of ξ^h in the sequel.

A normal almost contact metric structure is said to be *cosymplectic* if the 2-form φ and the 1-form η are both closed. It is known that the cosymplectic structure is characterized by (Blair [1])

(1.3) $$\nabla_k\varphi_j{}^i = 0 \text{ and } \nabla_k\eta^i = 0,$$

where ∇_k denotes the operator of covariant differentiation with respect to g_{ji}.

Now from equations (1.3) and the Ricci identity:

$$\nabla_k\nabla_j\eta^h - \nabla_j\nabla_k\eta^h = K_{kjt}{}^h\eta^t,$$

we find

(1.4) $$K_{kjt}{}^h\eta^t = 0,$$

from which, by contraction

(1.5) $$K_{jt}\eta^t = 0,$$

where $K_{kji}{}^h$ and K_{ji} are the curvature tensor and the Ricci tensor of a cosymplectic manifold M respectively.

From equations (1.3) and the Ricci identity:

$$\nabla_k\nabla_j\varphi_i{}^h - \nabla_j\nabla_k\varphi_i{}^h = K_{kjt}{}^h\varphi_i{}^t - K_{kji}{}^t\varphi_t{}^h,$$

we find

(1.6) $$K_{kjt}{}^h\varphi_i{}^t = K_{kji}{}^t\varphi_t{}^h,$$

from which, by contraction

(1.7) $$K_{jt}\varphi_i{}^t = -K_{tjis}\varphi^{ts},$$

where $\varphi^{ts} = g^{ti}\varphi_i{}^s$, g^{ti} being contravariant components of the metric tensor. Since

$$K_{tjis}\varphi^{ts} = K_{sijt}\varphi^{ts} = -K_{tijs}\varphi^{ts},$$

we have from (1.7)

(1.8) $$K_{it}\varphi_i{}^t + K_{it}\varphi_j{}^t = 0.$$

Since

$$K_{tjis}\varphi^{ts} = \frac{1}{2}(K_{tjis} - K_{sjit})\varphi^{ts} = -\frac{1}{2}K_{tsji}\varphi^{ts},$$

we also have from (1.7) and (1.8) that

(1.9) $$K_{tsji}\varphi^{ts}=2K_{jt}\varphi_i^t.$$

If the Ricci tensor of the cosymplectic manifold M is of the form $K_{ji}=\alpha(g_{ji}-\eta_j\eta_i)$, then M is called a *cosymplectic η-Einstein manifold.* By contraction, we have $\alpha=K/2n$, where K is the scalar curvature of M. Thus a cosymplectic η-Einstein manifold is characterized by

(1.10) $$K_{ji}=\frac{K}{2n}(g_{ji}-\eta_j\eta_i),$$

where the scalar curvature K is a constant by virtue of the identity

$$\nabla_t K_j^t=\frac{1}{2}\nabla_j K,$$

and from which, we obtain

(1.11) $$\nabla_k K_{ji}=0.$$

For example, if the φ-holomorphic sectional curvature of a cosymplectic manifold M is constant, then M is a cosymplectic η-Einstein manifold (Eum [2]).

If a cosymplectic η-Einstein manifold M with $K\neq 0$ admits a parallel vector v^h, then by the Ricci identity and (1.10), we have

(1.12) $$v_j=(\eta_t v^t)\eta_j.$$

Thus we have the following

LEMMA 1. *If a cosymplectic η-Einstein manifold with $K\neq 0$ admits a parallel vector field v^h, then $v^h=\lambda\eta^h$, where $\lambda=\eta_t v^t$.*

2. Contravariant C^*-analytic vector fields in a cosymplectic manifold

Recently, in a Sasakian manifold with the structure tensor φ of type (1, 1), I. Sato defined a contravariant C-analytic vector field u^i by $(\mathcal{L}(u)\varphi_j^i)\varphi_i^k=0$ (Sato [3]). Following this definition, we provide the following

DEFINITION. A vector field u in a cosymplectic manifold M is said to be *contravariant C^*-analytic*, if u satisfies

(2.1) $$(\mathcal{L}(u)\varphi_j^k)\varphi_k^h=0 \text{ and } \mathcal{L}(u)\eta^h=0.$$

where $\mathcal{L}(u)$ denotes the Lie derivation with respect to u.

The first equation of (2.1) is written by the covariant form

(2.2) $$\nabla_j u_i=\varphi_j^t\varphi_i^s\nabla_t u_s+(\nabla_j\lambda)\eta_i,$$

where we have put

(2.3) $$\lambda=u_t\eta^t.$$

The second equation of (2.2) is written by the form

(2.4) $\eta^t \nabla_t u^h = 0.$

Applying the operator ∇_k to (2.2) and transvecting it with g^{kj}, we obtain

(2.5) $$g^{kj}\nabla_k \nabla_j u_i = \varphi_i{}^t \varphi^{ks} \nabla_k \nabla_s u_t + g^{kj}\eta^s (\nabla_k \nabla_j u_s)\eta_i = -\frac{1}{2}\varphi_i{}^t \varphi^{ks} K_{kst}{}^p u_p + (\nabla^t \nabla_t \lambda)\eta_i$$

by virtue of the Ricci identity and (2.3).

Substituting (1.9) into (2.5), we obtain

(2.6) $$\nabla^t \nabla_t u^h + K_t{}^h u^t = (\nabla^t \nabla_t \lambda)\eta^h,$$

where ∇^t indicates the operator $g^{tk}\nabla_k$.

On the other hand, we have

$$\frac{1}{2}[\nabla_j u_i - \varphi_j{}^t \varphi_i{}^s \nabla_t u_s - (\nabla_j \lambda)\eta_i]\cdot[\nabla^j u^i - \varphi^{jb}\varphi^{ia}\nabla_b u_a - (\nabla^j \lambda)\eta^i]$$

$$= (\nabla^j u^i)(\nabla_j u_i) - \varphi^{jt}\varphi^{is}(\nabla_j u_i)(\nabla_t u_s) - (\nabla_j \lambda)(\nabla^j \lambda) - \frac{1}{2}(\eta^t \nabla_t u^j)(\eta^s \nabla_s u_j) + \frac{1}{2}\eta^t \eta^s (\nabla_t \lambda)(\nabla_s \lambda),$$

and from which, we obtain

$$\nabla_j[(\nabla^j u^i)u_i - \varphi^{jt}\varphi^{is}(\nabla_t u_s)u_i - \lambda(\nabla^j \lambda) - \frac{1}{2}\eta^t(\nabla_t \lambda)\eta^j + \frac{1}{2}\lambda\eta^t(\nabla_t \lambda)\eta^j$$

(2.7) $$= \frac{1}{2}[\nabla_j u_i - \varphi_j{}^t \varphi_i{}^s \nabla_t u_s - (\nabla_j \lambda)\eta_i][\nabla^j u^i - \varphi^{jt}\varphi^{is}\nabla_t u_s - (\nabla^j \lambda)\eta^i] + [\nabla^t \nabla_t u^i + K_t{}^i u^t - (\nabla^t \nabla_t \lambda)\eta^i]u_i - \frac{1}{2}\eta^t \nabla_t(\eta^r \eta^s \nabla_s u_r) + \frac{1}{2}\lambda\eta^t \nabla_t(\eta^r \eta^s \nabla_s u_r).$$

In this case, we assume that the manifold is compact. Then applying Green's theorem, we see that the integral over the whole space of the right hand member of (2.7) vanishes.

Taking account of this reason, (2.2) and (2.4), we have the following

THEOREM 1. *A necessary and sufficient condition for a vector field u^h in a compact cosymplectic manifold to be contravariant C^*-analytic is the following* (2.8) *and* (2.9).

(2.8) $\nabla^t \nabla_t u^i + K_t{}^i u^t - (\nabla^t \nabla_t \lambda)\eta^i = 0,$

(2.9) $\eta^t \nabla_t u^i = 0.$

Hereafter, we assume always that the cosymplectic manifold M is compact and we denote a contravariant C^*-analytic vector field by a *C^*-vector field* briefly in M.

The following lemma is well known (Yano [4]).

LEMMA 2. *A necessary and sufficient condition for a vector field v^h in M to be a Killing vector field is*

$$\nabla^t \nabla_t v^h + K_t{}^h v^t = 0 \text{ and } \nabla_t v^t = 0.$$

Transvecting $\varphi_i{}^h$ to (2.8) and (2.9) respectively, we have

(2.10) $$\nabla^t\nabla_t(\varphi_i{}^h u^i)+K_i{}^h(\varphi_t{}^i u^t)=0,\quad \eta^t\nabla_t(\varphi_i{}^t u^i)=0.$$

Thus we have the following

PROPOSITION 1. *If u^h is a C^*-vector field in M, then $\varphi_t{}^h u^t$ is also a C^*-vector field.*

Taking account of lemma 2 and (2.10), we obtain the following

PROPOSITION 2. *If u^h is a C^*-vector field and $\varphi_t{}^i\nabla_i u^t=0$, then $\varphi_t{}^h u^t$ is a Killing vector field.*

We investigate the relations between a Killing vector field and a C^*-vector field in M.

It is well known that if M admits a harmonic vector field η_i and a Killing vector field v^h, then $\eta_h v^h=$const. (Yano [4]). Therefore if v^h is a Killing vector field in M, then $\lambda=v^h\eta_h$ is a constant by virtue of the second equation of (1.3). Thus we have the following

PROPOSITION 3. *If M admits a Killing (or harmonic) vector field v^h, then $\eta_t v^t$ is a constant.*

On the other hand, if v^h is a Killing vector field, then

$$\eta^t\nabla_t v^i=-\eta^t\nabla^i v_t=-\nabla^i(\eta^t v_t)=0.$$

Thus by the theorem 1 and the lemma 2, we obtain the following

PROPOSITION 4. *If v^h is a Killing vector field in M, then v^h is a C^*-vector field.*

If v^h is a Killing vector field, then it satisfies (2.2) with $\nabla_j\lambda=0$ by virtue of the propositions 3 and 4, that is,

(2.11) $$\nabla_j v^k+\varphi_j{}^t\varphi_i{}^k\nabla_t v^i=0.$$

Transvecting $\varphi_k{}^h$ to (2.11), we have

(2.12) $$\nabla_j(\varphi_h{}^t v_t)-\nabla_h(\varphi_j{}^t v_t)=0$$

because of $\nabla_j v_i+\nabla_i v_j=0$ and $\nabla_k\lambda=0$.

If we assume that $\varphi^{ji}\nabla_j v_i=0$, that is

(2.13) $$\nabla_j(\varphi^{jt}v_t)=0,$$

then we see that $\varphi_h{}^t v_t$ is a harmonic vector field by virtue of (2.12).

Thus we have the following

LEMMA 3. *If v^h is a Killing vector field and $\varphi^{ji}\nabla_j v_i=0$, then $\varphi_j{}^t v_t$ is a harmonic vector field.*

The following lemma is well known (Yano [4]).

LEMMA 4. *A necessary and sufficient condition for a vector field w^h in M to be harmonic is* $\nabla^t\nabla_t w^h - K_t{}^h w^t = 0$.

Substituting $\varphi^{ht}v_t$ instead of w^h in lemma 4 and taking account of (1. 8), we easily see that

$$\varphi_{ih}\nabla^t\nabla_t v^i + K_{ti}\varphi_h{}^i v^t = 0. \tag{2. 14}$$

Transvecting (2. 14) with $\varphi_j{}^h$ and taking account of (1. 5) and proposition 2, we obtain

$$\nabla^t\nabla_t v_j - K_{tj}v^t = 0 \tag{2. 15}$$

under the assumption that v^h is a Killing vector field.

Taking account of lemmas 3, 4 and (2. 15), we have the following

LEMMA 5. *If v^h is a Killing vector field and $\varphi^{ji}\nabla_j v_i = 0$, then v^h is a harmonic vector field and v^h is a parallel vector field.*

Combining lemma 1 and lemma 5 and taking account of (1. 12), we have the following

PROPOSITION 5. *In a compact cosymplectic η-Einstein manifold with $K \neq 0$, if v^h is a Killing vector field such that $\varphi^{ji}\nabla_j v_i = 0$ and $v^h \neq c\eta^h$, then v^h vanishes.*

3. A decomposition of contravariant C^*-analytic vector fields in a compact cosymplectic η-Einstein manifold

In this section, we assume that M is a compact cosymplectic η-Einstein manifold with $K \neq 0$.

Let u^h be a contravariant C^*-analytic vector field, that is, a C^*-vector field in M.

In this case substituting (1. 10) into (2. 8), we obtain

$$g^{kt}[\nabla_k\nabla_t u^i - (\nabla_k\nabla_t\lambda)\eta^i] + \frac{K}{2n}(u^i - \lambda\eta^i) = 0, \tag{3. 0}$$

where $\lambda = u_t\eta^t$ and K is a non-zero constant.

Putting

$$u^i - \lambda\eta^i = v^i, \tag{3. 1}$$

we have

$$\eta_t v^t = 0. \tag{3. 2}$$

and (3. 0) is rewritten as

$$g^{kt}\nabla_k\nabla_t v^i + \frac{K}{2n}v^i = 0. \tag{3. 3}$$

Differentiating covariantly (3. 3), we obtain

(3.4) $$g^{tk}\nabla_j\nabla_k\nabla_t v^i + \frac{K}{2n}\nabla_j v^i = 0.$$

On the other hand, by the Ricci identity we have

(3.5) $$\nabla_j\nabla_k\nabla_t v^i = \nabla_k(\nabla_t\nabla_j v^i + K_{jts}{}^i v^s) + K_{jks}{}^i\nabla_t v^s - K_{jkt}{}^s\nabla_s v^i.$$

Substituting (3.5) into (3.4), contracting with respect to j and i and taking account of (1.10), (1.11) and (3.2), we obtain

(3.6) $$\nabla^t\nabla_t\nabla_i v^i + \frac{K}{n}\nabla_t v^t = 0.$$

Differentiating covariantly (3.6), using the Ricci identity and taking account of (1.10), we obtain

(3.7) $$\nabla^t\nabla_t\nabla^i\nabla_k v^k + \frac{K}{2n}\nabla^i\nabla_t v^t + \frac{K}{2n}\eta^i\eta_r\nabla^r\nabla_t v^t = 0.$$

On the other hand, a C^*-vector field u^h satisfies (2.2) and (2.4). Transvecting η^j to (2.2) and taking account of (2.4), we have

(3.8) $$\eta^j\nabla_j\lambda = 0.$$

Moreover by the Ricci identity and (1.4), we have

(3.9) $$\eta_r\nabla^r\nabla_t v^t = 0$$

by the help of (2.4) and (3.8).

Therefore (3.7) is rewritten as

(3.10) $$\nabla^t\nabla_t\nabla^i\nabla_k v^k + \frac{K}{2n}\nabla^i\nabla_t v^t = 0.$$

In this place, we put

(3.11) $$p^h = v^h + \frac{n}{K}\nabla^h\nabla_t v^t.$$

Differentiating covariantly (3.11) and taking account of (3.4) and (3.9), we see that

(3.12) $$\nabla^t\nabla_t p^h + K_t{}^h p^t = 0.$$

Moreover, we easily see that

(3.13) $$\nabla_t p^t = 0$$

by the help of (3.6).

Taking account of lemma 2, (3.12) and (3.13), we see that p^h is a Killing vector field.

Next, we put

(3.14) $$q_j = -\frac{n}{K}\varphi_j{}^t\nabla_t\nabla_i v^i.$$

Differentiating covariantly (3.14), we easily see that

(3.15) $$\nabla_t q^t = 0$$

because of $\nabla_i v^i$ is a scalar function.

Moreover, taking account of lemma 2, (3.9), (3.10) and (3.14), we see that q^h is a Killing vector field and $q_t\eta^t = 0$. Since

(3.16) $\varphi_t{}^h q^t = -\frac{n}{K}\nabla^h\nabla_i v^i,$

(3.11) is rewritten as

(3.17) $p^h = v^h - \varphi_t{}^h q^t.$

Substituting (3.2) into (3.17), we obtain

(3.18) $u^h - \lambda\eta^h = p^h + \varphi_t{}^h q^t,$

where p^h and q^h are both Killing vector fields.

If u^h is given, then $\lambda = u_t\eta^t$ is a fixed scalar function.

In this place, we can prove that the decomposition (3.18) of $u^h - \lambda\eta^h$ is unique. In fact, if

$$u^h - \lambda\eta^h = p^h + \varphi_t{}^h q^t = {}'p^h + \varphi_t{}^h{}' q^t,$$

where, $p^h, {}'p^h, q^h$ and ${}'q^h$ are all Killing vector fields and $q_t\eta^t = {}'q_t\eta^t = 0$, then we have

(3.19) $p^h - {}'p^h = \varphi_t{}^h({}'q^t - q^t),$

and from which,

(3.20) $\nabla_k(p_j - {}'p_j) = \nabla_k[\varphi_{tj}({}'q^t - q^t)].$

Since p^h and ${}'p^h$ are both Killing vector fields, the transvection to (3.20) with g^{kj} shows that

(3.21) $\varphi^{kj}\nabla_k({}'q_j - q_j) = 0,$

by virtue of lemma 2.

Taking account of the fact that ${}'q_j - q_j \neq c\eta_j$, (3.21) and proposition 5, we see that ${}'q_j - q_j$ vanishes, that is, ${}'q^h = q^h$, and consequently ${}'p^h = p^h$.

Thus, we have the following

THEOREM 2. *In a compact cosymplectic η-Einstein manifold with $K \neq 0$, a contravariant C^*-analytic vector field u^h is uniquely decomposed in the form*

$$u^h - \lambda\eta^h = p^h + \varphi_t{}^h q^t,$$

where p^h and q^h are both Killing vector fields, $\lambda = u_t\eta^t$ and $q_t\eta^t = 0$.

Combining proposition 1 and theorem 2, we have the following theorem 3 by virtue of (1.1)

THEOREM 3. *Let u^h be a contravariant C^*-analytic vector field in a compact cosymplectic η-Einstein manifold with $K \neq 0$ and let $v^h = \varphi_t{}^h u^t$. Then the vector field v^h is uniquely decomposed in the form*

$$v^h = p^h + \varphi_t{}^h q^t,$$

where p^h and q^h are both Killing vector field and $q_t\eta^t = 0$.

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Sung Kyun Kwan University