J. Korean Math. Soc.
Vol. 18, No. 1, 1981

BANACH SPACES OF LIPSCHITS FUNCTIONS

BY DONG SUN SHIN

1. Introduction

Although the notion of Lipschitz continuity is very old and Lipschitz functions have been studied for many years, the Banach space Lip (S, d) of Lipschitz functions has not developed until quite recently. For example, the following two problems are still open: If (S, d) is compact and infinite and $0<\alpha<1$, (1) is lip (S, d^α) isomorphic with c_0 and (2) is Lip (S, d^α) isomorphic with m (cf. [6])?

It is known that if (S, d) is an infinite compact subset of Euclidean space, the lip (S, d^α) $(0<\alpha<1)$ is isomorphic to c_0 and Lip (S, d^α) is isomorphic with m. And if (S, d) is compact and $0<\alpha<1$, then Lip (S, d^α) and lip (S, d^α) are isomorphic to subspaces of m and c_0, respectively. On the other hand, if (S, d) is a metric space with $\inf\{d(s,t): s\neq t\}=0$, then Lip (S, d) contains a subspace isomorphic to 1_∞ and lip (S, d^α) contains a complemented subspace isomorphic to c_0. (cf. [6]). The purpose of this paper is to study the relations among Lip, lip, c_0, m and $\mathcal{L}_p$ spaces and from our results on them to investigate extensions of compact and weakly compact operators related to lip (S, d^α).

2. Preliminaries

Let (S, d) be a metric space. A complex valued function f defined on S is said to be a *Lipschitz function* if there exists a constant K such that

$$|f(x)-f(y)|\leqq kd(x,y),\quad x, y\in S.$$

The smallest such constant K is called the *Lipschitz norm* of f which we shall denote by $\|f\|_d$.

Evidently,

$$\|f\|_d=\sup\{|f(x)-f(y)|/d(x,y) : x,y\in S,\ x\neq y\}.$$

For a complex valued function f defined on S which is bounded on S, the sup norm $\|f\|_\alpha$ of f is defined by

$$\|f\|_\infty=\sup\{|f(x)| : x\in S\}.$$

Received May 1, 1981.

The collection of all bounded Lipschitz functions on (S, d) will be denoted by Lip (S, d). It is well known that Lip (S, d) is a Banach space with the norm $\|\cdot\|$ defined by

$$\|f\| = \|f\|_\infty + \|f\|_d \quad f \in \text{Lip}\,(S, d).$$

An important subset of Lip (S, d) consists of all those f in Lip (S, d) which have the property that

$$|f(x) - f(y)|/d(x, y) \to 0 \text{ as } d(x, y) \to 0.$$

This can more precisely be stated as follows: For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$[|f(x) - f(y)|/d(x, y)] < \varepsilon, \text{ whenever } d(x, y) < \delta.$$

This set of functions in Lip (S, d) will be denoted by lip (S, d). lip (S, d) is a closed subspace of Lip (S, d). If $0 < \alpha \leqq 1$, then d^α is a metric on S, where $d^\alpha(x, y) = (d(x, y))^\alpha$. Thus we may consider Lip (S, d^α) and lip (S, d^α) for $0 < \alpha \leqq 1$.

It is known that if S is compact and $0 < \alpha \leqq 1$, then Lip (S, d^α) is isometrically isomorphic to the bidual lip $(S, d^\alpha)^{**}$ of lip (S, d^α) (cf. [6]).

Below we shall state a few definitions and lemmas necessary for our forthcoming arguments.

DEFINITION 1. The Banach space $m = 1_\infty$ is defined to be the space of all bounded sequences, $x = \{\alpha_n\}$ of complex numbers. The Banach space c_0 (a subspace of m) is defined to be the space of all sequences converging to zero.

DEFINITION 2. Let $1 \leqq p \leqq \infty$ and $1 \leqq \lambda < \infty$. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$ space if for every finite dimensional subspace B of X there exists a finite dimensional subspace C of X such that $C \supset B$ and the distortion $d(C, 1_p^n) \leqq \lambda$, where $n = \dim C$. ($d(C, 1_p^n) \leqq \lambda$ means that there exists an isomorphism T from C onto l_p^n such that $\|T\|\|T^{-1}\| \leqq \lambda$)

A Banach space is said to be an $\mathcal{L}_p$ space $1 \leqq p \leqq \infty$, if it is an $\mathcal{L}_{p,\lambda}$-space for some $1 \leqq \lambda < \infty$. It is well known that the $\mathcal{L}_p$ space generalize the $L_p(\mu)$ and $C(K)$ spaces.

DEFINITION 3. A Banach space is said to be *injective* if it is complemented in any Banach space containing it.

DEFINITION 4. Let X be a (real or complex) Banach space and B a closed linear subspace of X. B is said to be *quasi-complemented* if there exists a closed linear subspace D of X such that $B \cap D = \{0\}$, with $B + D$ dense in X (such a subspace D is called a quasi-complement for B).

DEFINITION 5. Let X be a Banach Space and $\{x_n\}$ be a sequence of elements in X. The series $\sum_{n=1}^{\infty} x_n$ is *w. u. c.* (*weakly unconditional convergent*) if for every permutation (k_n) of indices the series $\sum_{n=1}^{\infty} x_{k_n}$ converges weakly (may be that the limit element does not exist); that is, for any $y \in X^*$, $y(\sum_{n=1}^{l} x_{k_n})$ converges as $l \to \infty$.

The series $\sum_{n=1}^{\infty} x_n$ is *u. c.* (*unconditionally convergent*) if for every permutation (k_n) the series $\sum_{n=1}^{\infty} x_{k_n}$ converges.

DEFINITION 6 Let X, Y be Banach spaces. A linear operator T from X to Y is *compact* (resp. *weakly compact*) if and only if, for any bounded set B in X, $T(B)$ is relatively compact (resp. relatively weakly compact) in Y.

Let X, Y, Z be Banach spaces such that $Z \supset X$. Let T be a linear operator from X to Y. A linear operator $\tilde{T}$ from Z to Y is called an *extension* of T if and only if $\tilde{T}|_X = T$.

LEMMA 1. *If (S, d) is an infinite compact subset of Euclidean space, then* lip (S, d^α) $(0<\alpha<1)$ *is isomorphic with c_0 and* Lip (S, d^α) *is isomorphic with* $1_\infty = m$.

Proof. Refer to [2], [6].

LEMMA 2. *Let X be a Banach space whose dual X^* contains a subspace isomorphic to c_0. Then there exists a projection of X onto a subspace which is isomorphic to 1_1: therefore, X^* contains a subspace which is isomorphic to $m = 1_\infty$.*

Proof. Refer to [1].

LEMMA 3. *An infinite dimensional subspace X of m is complemented in m if and only if X is isomorphic to m.*

Proof. Refer to [7].

LEMMA 4.

(1) *X is an $\mathcal{L}_1$-space if and only if X^* is injective.*

(2) *Every infinite dimensional $\mathcal{L}_\infty$-space have a subspace isomorphic to c_0.*

(3) *A Banach space X is an $\mathcal{L}_p$ space $(1 \leq p \leq \infty)$ if and only if X^* is a $\mathcal{L}$ space where $\frac{1}{p}+\frac{1}{q}=1$ ($q=1$,* resp. ∞ *if $p=\infty$,* resp. 1).

(4) *Every injective space is an* $\mathcal{L}_\infty$ *space*
(5) *A Banach space* X *is an* $\mathcal{L}_\infty$ *space if and only if* X^{**} *is injective, and*
(6) *A dual Banach space* X (i. e., $X=Y^*$ *for some Banach space* Y) *is injective if and only if* X *is an* $\mathcal{L}_\infty$ *space.*

Proof. Refer to [9].

3. Theorems

THEOREM 1. $(1_\infty)^*$ *does not contain a subspace isomorphic to* c_0.

Proof. If $(1_\infty)^*$ contains a subspace isomorphic to c_0. then by Lemma 2 there exists a projection of 1_∞ onto a subspace which is isomorphic to 1_1, But by Lemma 3, 1_1 is isomorphic to 1_∞, which gives a contradiction. Hence $(1_\infty)^*$ does not contain a subspace isomorphic to c_0. This completes the proof.

COROLLARY 2. $(1_\infty)^*$ *is not injective.*

Proof. If $(1_\infty)^*$ is injective, then $(1_\infty)^*$ contains a subspace isomorphic to c_0 ([7]). By Theorem 1, this is a contradiction. Hence $(1_\infty)^*$ is not injective.

COROLLARY 3. *The following conditions are equivalent.*
(a) *There does not exist in* $(1_\infty)^*$ *a* w. u. c. *series which is not* u. c..
(b) *There does not exist in the space* $(1_\infty)^*$ *a* w. u. c. *series such that* $\inf_n \|x_n\|>0$.
(c) $(1_\infty)^*$ *does not contain a subspace isomorphic to* c_0.

Proof. It follows from Theorem 1 and [1].

THEOREM 4. *If* S *is an infinite compact subset of Euclidean space and* $0<\alpha<1$, *then*
(1) $(1_\infty)^*$ *does not contain a subspace isomorphic with* lip (S, d^α),
(2) Lip $(S, d^\alpha)^*$ *does not contain a subspace isomorphic with* c_0, *and*
(3) Lip $(S, d^\alpha)^*$ *does not contain a subspace isomorphic with* lip (S, d^α).

Proof. (1). By Lemma 1 and Theorem 1, lip (S, d^α) is isomorphic with c_0 and $(1_\infty)^*$ does not contain a subspace isomorphic with c_0. Therefore $(1^\infty)^*$ does not contain a subspace isomorphic with lip (S, d^α).

(2). By Lemma 1 and Theorem 1, Lip (S, d^α) is isomorphic with 1_∞ and $(1_\infty)^*$ does not contain a subspace which is isomorphic with c_0. Therefore, Lip $(S, d^\alpha)^*$ does not contain a subspace isomorphic with c_0.

(3). (3) is clear.

THEOREM 5. *Let* (S, d) *be an infinite compact subset of Euclidean space. Then* lip $(S, d^{\alpha})^*$ $(0<\alpha<1)$ *is a separable infinite dimensional* $\mathcal{L}_1$*-space.*

Proof Banach Space X is a separable infinite dimensional $\mathcal{L}_1$-space if and only if X^* is isomorphic 1_∞ ([9]). Since [lip $(S, d^{\alpha})^*]^*=$Lip (S, d^{α}) is isomorphic to 1_∞, (Lemma 1, [6]) lip $(S, d^{\alpha})^*$ is a separable infinite dimensional $\mathcal{L}_1$-space.

THEOREM 6. *Let* (S, d) *be compact and infinite and* $0<\alpha<1$. *Then the following assertions are equivalent*:

(a) Lip (S, d^{α}) *is injective,*
(b) Lip (S, d^{α}) *is an* $\mathcal{L}_\infty$ *space,*
(c) lip (S, d^{α}) *is an* $\mathcal{L}_\infty$ *space,*
(d) lip $(S, d^{\alpha})^*$ *is an* $\mathcal{L}_1$ *space, and*
(e) Lip (S, d^{α}) *is isomorphic with* L_∞.

Moreover, the above equivalent conditions (a)$\sim$(e) *implies that*
(f) Lip $(S, d^{\alpha})^*$ *does not contain a subspace isomorphic with* c_0,

Proof. (a)$\Leftrightarrow$(b) By Lemma 4, (6), the dual Banach space Lip $(S, d^{\alpha})=$ [lip $(S, d^{\alpha})^*$] * is an injective space if and only if Lip (S, d^{α}) is an $\mathcal{L}_\infty$ space.

(a)$\Leftrightarrow$(c) By Lemma 4, (5), Lip $(S, d^{\alpha})=$[lip $(S, d^{\alpha})^*]^*$ is injective if and only if lip (S, d^{α}) is $\mathcal{L}_\infty$ space. Thus (a) and (c) are equivalent.

(d)$\Leftrightarrow$(b) If lip $(S, d^{\alpha})^*$ is an $\mathcal{L}_1$ space, then by Lemma 4. (1), Lip (S, d^{α}) $=$[lip$(S, d^{\alpha})^*]^*$ is injective and hence Lip (S, d^{α}) is an $\mathcal{L}_\infty$ space ((a)$\Leftrightarrow$(b)).

Conversely, if Lip (S, d^{α}) is an $\mathcal{L}_\infty$ space, then Lip (S, d^{α}) is injective ((a)$\Leftrightarrow$(b)) and hence Lip (S, d^{α}) is an $\mathcal{L}_\infty$ space ((a)$\Leftrightarrow$(c)). Therefore lip $(S, d^{\alpha})^*$ is an $\mathcal{L}_1$ space (cf. Lemma 4, (3))

(d)$\Leftrightarrow$(e) If Lip (S, d^{α}) is isomorphic with l_∞, then Lip (S, d^{α}) is injective. Since (a) implies (d), lip $(S, d^{\alpha})^*$ is an $\mathcal{L}_1$ space.

Conversely, if lip $(S, d^{\alpha})^*$ is an $\mathcal{L}_1$ space, then, since lip $(S, d^{\alpha})^*$ is a separable infinite dimensional Banach space (cf: [6], (iii)). Lip $(S, d^{\alpha})=$ [lip$(S, d^{\alpha})^*]^*$ is isomorphic with l_∞ (cf. [9]). Now (e) $\Longrightarrow$ (f) is clear from Theorem 1.

We offer here an alternative proof for (a) $\Longrightarrow$ (f). We note that Lip (S, d^{α}) is isomorphic with a subspace of l_∞(cf. [6]). If Lip (S, d^{α}) is injcetive, then l_∞ is isomorphic with Lip $(S, d^{\alpha})\oplus Y$ for some subspace Y of l_∞. That is, $l_\infty\cong$Lip $(S, d^{\alpha})\oplus Y$. Therefore $(l_\infty)^*\cong$Lip $(S, d)^*\oplus Y^*$. Hence if Lip $(S, d^{\alpha})^*$ contains a subspace isomorphic with c_0, then $(l_\infty)^*$ also contains a subspace isomorphic with c_0, which contradict to Theorem 1.

THEOREM 7. *Let X be separable infinite dimensional $\mathcal{L}_1$ space. Then X^{**} does not contain a subspace isomorphic to c_0.*

Proof. By [9], X^* is isomorphic with l_∞ and hence $X^{**}=(l_\infty)^*$. Since $(l_\infty)^*$ does not contain a subspace isomorphic with c_0, X^{**} also contains no subspace isomorphic with c_0.

THEOREM 8. *Let (S,d) be infinite compact subset of Euclidean space. Then lip $(S,d^\alpha)^*$ $(0<\alpha<1)$ has a complemented subspace isomorphic to l_1.*

Proof. lip $(S,d^\alpha)^{**}=$Lip (S,d^α) is isomorphic to l_∞ ([6]) and $l_\infty\supset c_0$. By Lemma 2, Lip $(S,d)^*$ has a complemented subspace isomorphic to l_1.

Although c_0 [resp. lip (S,d^α)] is not complemented in l_∞ [resp. Lip (S,d^α) ([6]), c_0 is quasi-complemented in l_∞([11], [12]).
We will show that lip (S,d^α) is quasi-complemented in Lip (S,d^α).

THEOREM 9. *If (S,d) is an infinite compact subset of Euclidean space,* lip (S,d^α) $(0<\alpha<1)$ *is quasi-complemented in* Lip (S,d^α).

Proof. lip (S,d^α) [resp. Lip (S,d^α)] is isomorphic to c_0 (resp. l_∞) by Lemma 1. Since c_0 is quasi-complemented in l_∞, lip (S,d^α) is quasi-complemented in Lip (S,d^α).

New we turn our arguments on the extension of compact and weakly compact operators.

THEOREM 10. *Let (S,d) be compact and infinite. Let* $0<\alpha<1$. *If* Lip(S,d^α) *is injective, then for any Banach space Y and Z, the following are true:*

(1) *Every compact operator T from Y to* Lip (S,d^α) [resp. lip (S,d^α)] *has a compact extension $\tilde{T}$ from Z $(Z\supset Y)$ to* Lip (S,d^α) [resp. lip (S,d^α)].

(2) *Every compact operator T from* Lip (S,d^α)[resp. lip (S,d^α)] *to Y has a compact extension $\tilde{T}$ from $Z\supset$*Lip (S,d^α) [resp. $Z\supset$lip (S,d^α)] *to Y.*

(3) *Every weakly compact operator from* lip (S,d^α) *to Y has a weakly compact extension T from $Z\supset$*lip (S,d^α) *to Y.*

Proof. Since we assumed that Lip (S,d^α) is injective, by Theorem 5, Lip (S,d^α) and lip (S,d^α) are $\mathcal{L}_\infty$-space. The theorem follows from the Lindenstrauss' characterization of the $\mathcal{L}_\infty$-space (cf. [7], [9]).

COROLLARY 11. *Let (S,d) be a compact infinite subset of Euclidean space. Then* (1), (2), (3) *stated in Theorem* 10, *hold for* lip (S,d^α) *and* Lip (S,d^α) *for* $0<\alpha<1$.

Proof. If (S, d) is a compact infinite subset of Euclicean space, then Lip (S, d^{α}) is isomorphic with l_{∞} and Lip (S, d^{α}) is injective (Th. 6). Therefore, by Theorem 10 our corollary is true.

We shall prove]here that the counterpart of (3) in Theorem 10 does not hold in general.

THEOREM 12. *Let (S, d) be a compact infinite metric space. We assume that* $\inf\{d(s,t) \mid s \neq t\}=0$. *Then there is a weakly compact operator T from a Banach space Y to* lip (S, d^{α}) *which does not have any weakly compact extension $\tilde{T}$ from a certain Banach space Z $(Z \supset Y)$ to* lip (S, d^{α}).

Proof. By the result obtained in [6], if $\inf\{d(s,t) \mid s \neq t\}=0$, then lip (S, d^{α}) contains a subspace isomorphic with c_0. Let T be the formal identity operator from 1_2 *to* c_0 ; *that is,* T maps the sequence $(x_1, x_2, x_3, \cdots\cdots)$ in 1_2 to the same sequence in c_0. We can regard T as a linear operator from 1_2 to lip (S, d^{α}). Now since 1_2 is reflexive, it follows that T is weakly compact.

Let $Z=1_{\infty}$ be the space of all bounded sequences. Then $1_{\infty} \supset 1_2$. This is true since 1_2 is separable and separable Banach space can be imbedded in 1_{∞} [10].

Now the weakly compact operator T from 1_2 to lip (S, d^{α}) does not have a weakly compact extension $\tilde{T}$ from $Z=1_{\infty}$ to lip (S, d^{α}). In fact, if there is an extension $\tilde{T}$, then $\tilde{T}$ has to map a weakly convergent sequence in Z to a norm convergent sequence in lip (S, d^{α}). However, if we let $\{e_i\}^{\infty}{}_{i=1}$ be the natural basis of 1_2, then $\{e_i\}^{\infty}{}_{i=1}$ converges weakly to 0 in Z. While $\{Te_i\}^{\infty}{}_{i=1}$ does not converge in the norm of lip (S, d^{α}). Thus T has no weakly compact extension.

THEOREM 13. *Let T be an operator from a $C(K)$ space ($K=$The Stcne-Cheh compactification of the integers) into a $1_{\infty}{}^*$. Then T is weakly compact.*

Proof. If T is an operator from a $C(K)$ into a Banach space X which does not have a subspace isomorphic to c_0, T is weakly compact [10].

By Theorem 1, T is weakly compact.

References

1. C. Bessaga and A. Pelezynski, *On bases and unconditional convergence of series in Banach space*, Studia Math., TXVII (1958)
2. R. Bonic, J. Framton and A. Tromba, *Λ-manifold*, *J. Funct. Analysis*, **3** (1969)

3. M. M. Day, *Normed linear space*, Springer Verlag (1958)
4. K. Deleeuw, *Banach spaces of Lip. function*, Studia Math. (1961)
5. Dunford and Schwarz, *Linear Operator I, II Interscience* (1967)
6. Z. A. Johnson,
   (i) *Banach spaces of Lip. function and Vector Valued Lip. function.* Tran. Amer. Math. Soc. **148** (1970)
   (ii) *Lip. function spaces for arbitrary metrics*, Bull. Amer. Math. Soc. **78** (1972)
   (iii) *Lip. spaces*, Pacific Jour. of Math. **51** (1974)
7. J. Lindenstrauss,
   (i) *Extension of compact operators*, Memo, A. M. S. (1964)
   (ii) *On complemented subspace of m.* Israel Jour. of Math. (1968)
8. J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in L_p spaces and Application*, Studia Math. (1968)
9. J. Lindenstrauss and H. P. Rosenthal, *The $\mathcal{L}_\infty$ spaces*, Israel Jour of Math. **7** (1967)
10. J. Lindenstrauss and L. Tzafrivi, *Classical Banach spaces I, II, Springer-Verlag* (1979)
11. H. P. Resenthal, *Quasi-complemented of Banach spaces, with an appendix on compactness of operators from L^p (μ) to $L^r(v)$*, J. Functional Analysis (4) 1969.
12. R. Whitley, *Proiecting m onto c_0*, Monthly 73, A. M. S. (1966)

Ewha Womans University