J. Korean Math. Soc.
Vol. 18, No. 1, 1981

ON THE RELATIONS OF TWO EINSTEIN'S 4-DIMENSIONAL UNIFIED FIELD THEORIES

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1. Introduction

A. *Two Einstein's 4-dimensional unified field theories*

In the usual Einstein's 4-dimensional unified field theory (g-UFT), the generalized Riemannian space X_4 referred to a real coordinate transformations $x^\nu \leftrightarrow \bar{x}^\nu$, for which

(1.1) $$\mathrm{Det}\left(\left(\frac{\partial \bar{x}}{\partial x}\right)\right) \neq 0,$$

is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$ * :

(1.2) $$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

where the matrices $g_{\lambda\mu}$ and $h_{\lambda\mu}$ are assumed to be of rank 4. We may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

(1.3) $$h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu.$$

In our 4-dimensional g-UFT, we use both $h_{\lambda\mu}$ and $h^{\lambda\nu}$ as tensors for raising and/or lowering indices of all tensors defined in X_4 in the usual manner.

In our subsequent considerations, the following densities, scalars, and tensors are frequently used:

(1.4)a $$\mathfrak{g} = \mathrm{Det}\, g_{\lambda\mu}\,,\quad \mathfrak{h} = \mathrm{Det}\, h_{\lambda\mu}\,,\quad \mathfrak{k} = \mathrm{Det}\, k_{\lambda\mu}\,,$$

(1.4)b $$g = \frac{\mathfrak{g}}{\mathfrak{h}},\quad k = \frac{\mathfrak{k}}{\mathfrak{h}};$$

(1.4)c $$^{(0)}k_\lambda{}^\nu = \delta_\lambda^\nu,\quad ^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu,\quad (p=1, 2, \cdots);$$

(1.4)d $$4K = k_{\lambda\mu} k^{\lambda\mu} = -{}^{(2)}k_\alpha{}^\alpha.$$

If $E^{\omega\mu\lambda\nu}$ and $e_{\omega\mu\lambda\nu}$ are indicators of tensor density of weight $+1$ and -1 respectively, they satisfy ([2], [4])

(1.5)a $$E_{\omega\mu\lambda\nu} = \mathfrak{h} e_{\omega\mu\lambda\nu},$$

(1.5)b $$\mathfrak{h} e^{\omega\mu\lambda\nu} = E^{\omega\mu\lambda\nu},$$

Received April 27, 1981.

(*) All considerations in the present paper are restricted to the 4-dimensional cases. Therefore, Greek indices take values 1, 2, 3, 4 and follow the summation convention throughout the present paper.

(1.6) $$E^{\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_4}e_{\beta_1\cdots\beta_p\alpha_{p+1}\cdots\alpha_4}=p!\,(4-p)\,!\,\delta^{\alpha_1\cdots\alpha_p}_{[\beta_1\cdots\beta_p]},\quad (p=0,1,2,3,4).$$

Also we may easily verify the following relations ([4]):

(1.7) a $$64\mathfrak{k}=(E^{\omega\mu\lambda\nu}\,k_{\omega\mu}k_{\lambda\nu})^2,$$

(1.7) b $$\mathfrak{g}=\mathfrak{h}+2\mathfrak{h}K+\mathfrak{k},\ \text{ or }\ g=1+2K+k;$$

(1.8) $$^{(p+4)}k_\lambda{}^\nu+2K\,^{(p+2)}k_\lambda{}^\nu+k\,^{(p)}k_\lambda{}^\nu=0,\quad (p=0,1,2,\cdots).$$

On the other hand, Einstein's 4-dimensional $^*g^{\lambda\nu}$*-unified field theory* ($^*g$-UFT) *in the same space* X_4 *is* based on the basic real tensor $^*g^{\lambda\nu}$ defined by

(1.9) $$g_{\lambda\mu}g^{\lambda\nu}=\delta_\mu{}^\nu.$$

It may also be decomposed into its symmetric part $^*h^{\lambda\nu}$ and its skew-symmetric part $^*k^{\lambda\nu}$:

(1.10) $$^*g^{\lambda\nu}={}^*h^{\lambda\nu}+{}^*k^{\lambda\nu}.$$

Since $\mathrm{Det}((^*h^{\lambda\nu}))\neq 0$ ([4], pp. 41), we may define a unique tensor $^*h_{\lambda\mu}$ by

(1.11) $$^*h^{\lambda\nu}\,{}^*h_{\lambda\mu}=\delta_\mu{}^\nu.$$

In our 4-dimensional $^*g$-UFT *we use both* $^*h^{\lambda\nu}$ *and* $^*h_{\lambda\mu}$, *in stead of* $h^{\lambda\nu}$ *and* $h_{\lambda\mu}$, *as tensors for raising and/or lowering indices of all tensors defined in* X_4 *in the usual manner*, with the exception of the tensors $g_{\lambda\mu}$, $h_{\lambda\mu}$, and $k_{\lambda\mu}$ in order to avoid the notational confusion. We then have, for example,

(1.12) a $$^*k_{\lambda\mu}={}^*k^{\rho\sigma}\,{}^*h_{\lambda\rho}\,{}^*h_{\mu\sigma},\quad {}^*g_{\lambda\mu}={}^*g^{\rho\sigma}\,{}^*h_{\lambda\rho}\,{}^*h_{\mu\sigma},$$

so that

(1.12) b $$^*g_{\lambda\mu}={}^*h_{\lambda\mu}+{}^*k_{\lambda\mu}.$$

In 4-dimensional $^*g$-UFT it may also be verified that the densities and scalars, defined by

(1.13) a $$^*\mathfrak{g}=\mathrm{Det}((^*g_{\lambda\mu})),\quad {}^*\mathfrak{h}=\mathrm{Det}((^*h_{\lambda\mu})),\quad {}^*\mathfrak{k}=\mathrm{Det}((^*k_{\lambda\mu}));$$

(1.13) b $$4\,{}^*K={}^*k_{\alpha\beta}\,{}^*k^{\alpha\beta}=-\,^{(2)*}k_\alpha{}^\alpha,$$

satisfy the following relation:

(1.14) $$^*\mathfrak{g}={}^*\mathfrak{h}+2\,{}^*\mathfrak{h}\,{}^*K+\mathfrak{k}.$$

B. *Purpose*

The relations of two 4-dimensional unified field theories were primarily studied by one of the authors ([1], pp. 1304), using a lengthy and complicated representation of the tensor field $^*g_{\lambda\mu}$ in terms of $g_{\lambda\mu}$ ([4], pp. 8 ; [1], pp. 1304). The purpose of the present paper is to derive more refined and simplified representations of $^*g_{\lambda\mu}$ than Hlavatý and Chung's results in the first, to find several useful relations satisfied by scalars defined by (1.4) and (1.13), and finally to investigate relations between two 4-dimensional unified field theories g-UFT and $^*g$-UFT. These topics will be studied for all classes and all possible indices of inertia.

2. The tensors $^*g_{\lambda\mu}$ and $^*g^{\lambda\nu}$

REMARK (2.1). All considerations in the present paper are dealt for all classes and all possible indices of inertia and restricted to 4-dimensional case.

THEOREM (2.2). *We have*

$$^*h^{\lambda\nu}=\frac{1}{g}\{(1+2K)h^{\lambda\nu}+{}^{(2)}k^{\lambda\nu}\}, \tag{2.1a}$$

$$^*k^{\lambda\nu}=\frac{1}{g}\{(1+2K)k^{\lambda\nu}+{}^{(3)}k^{\lambda\nu}\}. \tag{2.1b}$$

Proof. Hlavatý already obtained (2.1)a and a representation of $^*k^{\lambda\nu}$ in X_4 as follows ([4], pp. 8):

$$^*k^{\lambda\nu}=\frac{1}{\mathfrak{g}}(\mathfrak{h}k^{\lambda\nu}+\frac{k}{2}\sqrt{\mathfrak{k}}\,E^{\omega\mu\lambda\nu}\,k_{\omega\mu}),\quad \kappa=\operatorname{sgn} E^{\omega\mu\lambda\nu}k_{\omega\mu}k_{\lambda\nu}. \tag{2.2}$$

Employing (1.6), (1.5)a, (1.7)a, and (1.4)c, the second term of (2.2) can be simplified as in the following way:

$$\begin{aligned}\frac{\kappa}{2}\sqrt{\mathfrak{k}}\,E^{\omega\mu\lambda\nu}k_{\omega\mu}&=\frac{1}{16}(E^{\alpha\beta\gamma\delta}\,k_{\alpha\beta}\,k_{\gamma\delta})\,E_{\omega\mu\rho\sigma}\,k^{\omega\mu}h^{\rho\lambda}h^{\sigma\nu}\\ &=\frac{\mathfrak{h}}{16}\,E^{\alpha\beta\gamma\delta}e_{\omega\mu\rho\sigma}k_{\alpha\beta}k_{\gamma\delta}k^{\omega\mu}h^{\rho\lambda}h^{\sigma\nu}\\ &=\frac{\mathfrak{h}}{16}\,4!\;k_{[\omega\mu}k_{\rho\sigma]}k^{\omega\mu}k^{\rho\lambda}k^{\sigma\nu}\\ &=\mathfrak{h}(2Kk^{\lambda\nu}+{}^{(3)}k^{\lambda\nu}).\end{aligned} \tag{2.3}$$

(2.1)b is obtained by substituting (2.3) into (2.2).

REMARK (2.3). The result (2.1)b is more refined expression than Hlavatý's result (2.2). It may also be obtained from the n-dimensional representation of $^*k^{\lambda\nu}$ ([3], (4.6)b).

THEOREM (2.4). *In addition to* (2.1), *we have*

$$^*h_{\lambda\mu}=h_{\lambda\mu}-{}^{(2)}k_{\lambda\mu}, \tag{2.4a}$$

$$^*k_{\lambda\mu}=k_{\lambda\mu}-{}^{(3)}k_{\lambda\mu}, \tag{2.4b}$$

so that

$$^*g_{\lambda\mu}=g_{\lambda\mu}-{}^{(2)}k_{\lambda\mu}-{}^{(3)}k_{\lambda\mu}. \tag{2.4c}$$

Proof. The proof of (2.4)a is given in [1], p. 8. Substituting (2.1)b and (2.4)a into (1.12)a and making use of (1.4)c, (1.8), and (1.7)b, we have (2.4)b as in the following way:

$$\begin{aligned}^*k_{\lambda\mu}&={}^*k^{\rho\sigma}{}^*h_{\rho\lambda}{}^*h_{\sigma\mu}\\ &=\frac{1}{g}\{(1+2K)k_{\lambda\mu}-(1+4K)\,{}^{(3)}k_{\lambda\mu}-(1-2K)\,{}^{(5)}k_{\lambda\mu}+{}^{(7)}k_{\lambda\mu}\}\\ &=k_{\lambda\mu}-{}^{(3)}k_{\lambda\mu}.\end{aligned}$$

(2. 4) c is a direct result of (2. 4) a, (2. 4) b, and (1. 12) b.

REMARK (2. 5). We note that (2. 4) b is more refined than Chung's result ([1], p. 8).

$$ {}^*k_{\lambda\mu}=\frac{1}{g}\{(1-K)k_{\lambda\mu}-2(1+K)\,{}^{(3)}k_{\lambda\mu}+\frac{\kappa}{2}\sqrt{\mathfrak{k}}\,(e_{\lambda\mu\alpha\beta}k^{\alpha\beta} $$
$$ +2e_{\alpha\beta\gamma[\lambda}\,{}^{(2)}k_{\mu]}{}^{\alpha}k^{\beta\gamma}+e_{\alpha\beta\gamma\delta}k^{\alpha\beta\,(2)}k_{[\lambda}{}^{\gamma\,(2)}k_{\mu]}{}^{\delta})\}. $$

3. Relations of two 4-dimensional unified field theories

In this section, we derive representations of the determinants $^*\mathfrak{h}$ and $^*\mathfrak{k}$ first of all and investigate the relations of two 4-dimensional g-UFT and $^*g$-UFT.

THEOREM (3. 1). *The scalars* ${}^{(p)}k_{\alpha}{}^{\alpha}$ *may be given as*

(3. 1) a $\quad {}^{(p)}k_{\alpha}{}^{\alpha}=0,$ *when* p *is an odd integer,*

(3. 1) b $\quad {}^{(o)}k_{\alpha}{}^{\alpha}=4,$

(3. 1) c $\quad {}^{(2)}k_{\alpha}{}^{\alpha}=-4K,$

(3. 1) d $\quad {}^{(4)}k_{\alpha}{}^{\alpha}=4(2K^2-k),$

(3. 1) e $\quad {}^{(6)}k_{\alpha}{}^{\alpha}=4K(3k-4K^2),$

(3. 1) f $\quad {}^{(8)}k_{\alpha}{}^{\alpha}=4(8K^4-8kK^2+k^2),$

(3. 1) g $\quad {}^{(10)}k_{\alpha}{}^{\alpha}=4K(-16K^4+20kK^2-5k^2),$

(3. 1) h $\quad {}^{(12)}k_{\alpha}{}^{\alpha}=4(32K^6-48kK^4+18k^2K^2-k^3).$

Proof. (3. 1) are direct results of (1. 8).

In our further considerations, we use the following scalars:

(3. 2) $\quad M_p=E^{\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_4}E^{\beta_1\cdots\beta_p\beta_{p+1}\cdots\beta_4}\,{}^{(2)}k_{\alpha_1\beta_1}\cdots$
$\qquad\cdots{}^{(2)}k_{\alpha_p\beta_p}h_{\alpha_{p+1}\beta_{p+1}}\cdots h_{\alpha_4\beta_4},\quad (p=0, 1, 2, 3, 4);$

(3. 3) $\quad K_o=1,\quad K_p={}^{(2)}k_{[\alpha_1}{}^{\alpha_1\,(2)}k_{\alpha_2}{}^{\alpha_2}\cdots{}^{(2)}k_{\alpha_p]}^{\alpha_p},\quad (p=1, 2, 3, 4);$

(3. 4) $\quad W(p, q, r, s)=4!\,{}^{(p)}k_{[\alpha}{}^{\alpha\,(q)}k_{\beta}{}^{\beta\,(r)}k_{\gamma}{}^{\gamma\,(s)}k_{\delta]}{}^{\delta},\quad (p, q, r, s:$ integers).

THEOREM (3. 2). *We have*

(3. 5) $$M_p=p!\,(4-p)!\,\mathfrak{h}K_p,\quad (p=0, 1, 2, 3, 4).$$

Proof. Using (1. 5) a and (1. 6), the above result may be obtained as in the following way:

$$ M_p=E^{\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_4}E_{\beta_1\cdots\beta_p\alpha_{p+1}\cdots\alpha_4}\,{}^{(2)}k_{\alpha_1}{}^{\beta_1}\cdots{}^{(2)}k_{\alpha_p}{}^{\beta_p} $$
$$ =\mathfrak{h}E^{\alpha_1\cdots\alpha_p\alpha_{p+1}\dots\alpha_4}\;e_{\beta_1\cdots\beta_p\alpha_{p+1}\cdots\alpha_4}\,{}^{(2)}k_{\alpha_1}{}^{\beta_1}\cdots{}^{(2)}k_{\alpha_p}{}^{\beta_p} $$
$$ =p!\,(4-p)!\,\mathfrak{h}K_p. $$

In particular, for particular values of p, we may obtain the following table in virtue of (3. 1):

(3.6)

p	K_p	M_p
0	1	$4!\mathfrak{h}$
1	$-4K$	$-4!\mathfrak{h}K$
2	$4K^2+2k$	$8\mathfrak{h}(2K^2+k)$
3	$-4kK$	$-4!\mathfrak{h}kK$
4	k^2	$4!\mathfrak{h}k^2$

THEOREM (3.3). *When p, q, r, s are odd integers, we have*

$$W(p, q, r, s) = -6^{(p+q+r+s)}k_\alpha{}^\alpha + {}^{(p+q)}k_\alpha{}^\alpha\, {}^{(r+s)}k_\beta{}^\beta \tag{3.7}$$
$$+ {}^{(p+r)}k_\alpha{}^\alpha\, {}^{(q+s)}k_\beta{}^\beta + {}^{(p+s)}k_\alpha{}^\alpha\, {}^{(q+r)}k_\beta{}^\beta.$$

Proof. Using the skew-symmetry of ${}^{(p)}k_\lambda{}^\nu$ when p is an odd integer, the proof of (3.7) follows from (1.4)c and (3.4).

In particular, in virtue of (3.1) we have

$$W(1,1,1,1)=24k, \qquad W(1,1,1,3)=-24kK,$$
$$W(1,1,3,3)=8k(2K^2+k), \quad W(1,3,3,3)=-24k^2K, \tag{3.8}$$
$$W(3,3,3,3)=24k^3.$$

THEOREM (3.4). *We have*

$$^*K=K. \tag{3.9}$$

Proof. Using (1.13)b, (2.1)b, (2.4)b, (1.4)c, (3.1), and (1.7)b, the assertion of the theorem may be obtained as in the following way:

$$4^*K={}^*k_{\lambda\mu}{}^*k^{\lambda\mu}=\frac{1}{g}(k_{\lambda\mu}-{}^{(3)}k_{\lambda\mu})\{(1+2K)k^{\lambda\mu}+{}^{(3)}k^{\lambda\mu}\}=4K.$$

Now, we are ready to prove the following important relations:

THEOREM (3.5). *We have*

(3.10)a $\quad {}^*\mathfrak{h}=\mathfrak{h}g^2,$

(3.10)b $\quad {}^*\mathfrak{k}=\mathfrak{k}g^2,$

so that

(3.10)c $\quad {}^*\mathfrak{g}=\mathfrak{g}g^2.$

Proof. In virtue of (2.4)a, (3.2), (3.6), and (1.7)b, the first relation (3.10)a may be derived as in the following way:

$$4!^*\mathfrak{h} = E^{\omega\mu\lambda\nu}E^{\alpha\beta\gamma\delta*}h_{\omega\alpha}{}^*h_{\mu\beta}{}^*h_{\lambda\gamma}{}^*h_{\nu\delta}$$
$$= E^{\omega\mu\lambda\nu}E^{\alpha\beta\gamma\delta}(h_{\omega\alpha} - {}^{(2)}k_{\omega\alpha})(h_{\mu\beta} - {}^{(2)}k_{\mu\beta})(h_{\lambda\gamma} - {}^{(2)}k_{\lambda\gamma})(h_{\nu\delta} - {}^{(2)}k_{\nu\delta})$$
$$= M_0 - 4M_1 + 6M_2 - 4M_3 + M_4 = 4!\mathfrak{h}g^2.$$

On the other hand, using (1.5)a, (1.6), (2.4)b, (3.4), (3.8), (1.4)b, and (1.7)b, we have

$$4!^*\mathfrak{k} = E^{\omega\mu\lambda\nu}E^{\alpha\beta\gamma\delta*}k_{\omega\alpha}{}^*k_{\mu\beta}{}^*k_{\lambda\gamma}{}^*k_{\nu\delta}$$
$$= \mathfrak{h}E^{\omega\mu\lambda\nu}e_{\alpha\beta\gamma\delta}{}^*k_\omega{}^{\alpha*}k_\mu{}^{\beta*}k_\lambda{}^{\gamma*}k_\nu{}^\delta = 4!\mathfrak{h}^*k_{[\alpha}{}^{\alpha*}k_\beta{}^{\beta*}k_\gamma{}^{\gamma*}k_{\delta]}{}^\delta$$
$$= 4!\mathfrak{h}(k_{[\alpha}{}^\alpha - {}^{(3)}k_{[\alpha}{}^\alpha)(k_\beta{}^\beta - {}^{(3)}k_\beta{}^\beta)(k_\gamma{}^\gamma - {}^{(3)}k_\gamma{}^\gamma)(k_{\delta]}{}^\delta - {}^{(3)}k_{\delta]}{}^\delta)$$
$$= \mathfrak{h}\{W(1,1,1,1) - 4W(1,1,1,3) + 6W(1,1,3,3) - 4W(1,3,3,3) + W(3,3,3,3)\}$$
$$= 4!\mathfrak{k}(1+2K+k)^2 = 4!\mathfrak{k}g^2,$$

which proves the second statement (3.10)b. The last relation (3.10)c may be obtained by substituting for $^*\mathfrak{h}$, $^*\mathfrak{k}$, and $^*K$ from (3.10)a, (3.10)b, and (3.9) into (1.14).

Since $g \neq 0$, the preceding two theorems give us the following important theorem concerning the classification of two 4-dimensional unified field theories:

THEOREM (3.6). *The classification of the tensor field* $g_{\lambda\mu}$ *in* g-UFT *is identical to that of the tensor field* $^*g_{\lambda\mu}$ *in* $^*g$-UFT.

In addition to the relation stated in Theorem (3.6) the following theorem, proved in [1], p. 1307, gives the complete relationship between two 4-dimensional Einstein's unified field theories g-UFT and $^*g$-UFT.

THEOREM (3.7). *The signature of the tensor* $h_{\lambda\mu}$ *in* g-UFT *is identical to that of the tensor* $^*h_{\lambda\mu}$ *in* $^*g$-UFT.

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