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A NOTE ON η-CONTINUOUS FUNCTIONS

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1. Introduction

In [1] a new class of functions in topological spaces, called η-continuous, is introduced and some sufficient conditions for a weakly-continuous function to be η-continuous are established. Among others, the following result is important:

THEOREM A. *An open weakly-continuous function is η-continuous.*
The main purpose of this note is to give an improvement of the preceding theorem and to answer the following question posed by Rose [10]:

QUESTION B. *Is every almost-open weakly-continuous function necessarily almost-continuous in the sense of Singal?*

2. Preliminaries

Throughout this note spaces mean topological spaces and functious are not necessarily continuous. Let S be a subset of a space. The closure of S and the interior of S are denoted by $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$, respectively. A subset S is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(S))=S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S))=S$). The family of all regular open sets of a space X is denoted by $RO(X)$. A function $f: X\to Y$ is said to be *δ-continuous* [9] (resp. *almost-continuous* [11], *θ-continuous* [2], *weakly-continuous* [6]) if for each $x\in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathrm{IntCl}(U))\subset \mathrm{Int}(\mathrm{Cl}(V))$ (resp. $f(U)\subset \mathrm{Int}(\mathrm{Cl}(V))$, $f(\mathrm{Cl}(U))\subset \mathrm{Cl}(V)$, $f(U)\subset \mathrm{Cl}(V)$). Henceforth, "weakly-continuous" will be denoted by "*w. c.*".

DEFINITION 2.1. A function $f: X\to Y$ is said to be *η-continuous* [1], if for any $U, V\in RO(Y)$,

(a) $f^{-1}(V)\subset \mathrm{Int}\ (\mathrm{Cl}(f^{-1}(V))$ and

(b) $\mathrm{Int}(\mathrm{Cl}(f^{-1}(U\cap V)))=\mathrm{Int}(\mathrm{Cl}(f^{-1}(U))\cap \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$.

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In [9] the present author showed that the concepts of continuity and δ-continuity are independent of each other and the concept of δ-continuity is strictly stronger than that of almost-continuity. It is obvious that θ-continuity implies weak-continuity. However, the converse is not true [7, Example]. In [1] it is shown that almost-continuity implies η-continuity and η-continuity implies θ-continuity. None of these implications is reversible, as the following examples show.

EXAMPLE 2.2. Let $X=\{a,b,c,d\}$ and $\sigma=\{\phi, \{d\}, \{a,c\}, \{a,c,d\}, X\}$. Let $Y=\{x,y,z\}$ and $\tau=\{\phi, \{x\}, \{z\}, \{x,z\}, Y\}$. Define a function $f: (X,\sigma)\to(Y,\tau)$ as follows:

$$f(a)=x,\ f(b)=f(c)=y \text{ and } f(d)=z.$$

Then, f is η-continuous, but it is not almost-continuous.

EXAMPLE 2.3. Let X be the set of real numbers and σ be the countable complement topology for X. Let (Y,τ) be the space in Example 2.2. Define a function $f: (X,\sigma)\to(Y,\tau)$ as follows:

$$f(a)=x \text{ if } a \text{ is rational and } f(a)=y \text{ if } a \text{ is irrational.}$$

Then, f is θ-continuous, but it is not η-continuous.

3. Weakly almost-open functions

In [10], Rose defined a function $f: X\to Y$ to be *almost-open* if $f(U)\subset \mathrm{Int}(\mathrm{Cl}(f(U))$ for every open set U of X and showed that a function $f: X\to Y$ is almost-open if and only if $f^{-1}(\mathrm{Cl}(V))\subset \mathrm{Cl}(f^{-1}(V))$ for every open set V of Y. As a slight generalization of such functions, we have

DEFINITION 3.1. A function $f: X\to Y$ is said to be *weakly almost open* if $f^{-1}(\mathrm{Cl}(V))\subset \mathrm{Cl}(f^{-1}(V))$ for every $V\in RO(Y)$.

A space X is said to be *extremally disconnected* [12] if the closure of every open set in X is open.

LEMMA 3.2. *If a space Y is extremally disconnected, then every function $f: X\to Y$ is weakly almost-open.*

Proof. Let $V\in RO(Y)$. Then, we have $\mathrm{Cl}(V)=\mathrm{Int}(\mathrm{Cl}(V))=V$ and hence $f^{-1}(\mathrm{Cl}(V))=f^{-1}(V)\subset \mathrm{Cl}(f^{-1}(V))$.

The following example shows that a weakly almost-open function is not necessarily almost-open even if the range is extremally disconnected.

EXAMPLE 3.3. Let (X, σ) be the space of Example 2.2. Let $Y=\{x, y, z\}$ and $\tau=\{\phi, \{y\}, \{z\}, \{x, y\}, \{y, z\}, Y\}$. Define a function $f:(X, \sigma) \rightarrow (Y, \tau)$ as follows: $f(a)=x$, $f(b)=f(d)=y$ and $f(c)=z$. Then, the space (Y, τ) is extremally disconnected, and by Lemma 3.2 f is weakly almost-open. However, f is not almost-open.

THEOREM 3.4. *If a function $f: X \rightarrow Y$ is weakly almost-open and w. c., then $f^{-1}(V) \in RO(X)$ for every $V \in RO(Y)$.*

Proof. Let F be any regular closed set of Y. Since $\mathrm{Int}(F) \in RO(Y)$, by using Theorem 4 of [8] we have $\mathrm{Cl}(f^{-1}(\mathrm{Int}(F)))=f^{-1}(\mathrm{Cl}(\mathrm{Int}(F)))=f^{-1}(F)$. Hence $f^{-1}(F)$ is closed in X. Since f is w. c., $f^{-1}(\mathrm{Int}(F)) \subset \mathrm{Int}(f^{-1}(F))$ [6, Theorem 1]. Thus, we obtain

$$f^{-1}(F)=\mathrm{Cl}(f^{-1}(\mathrm{Int}(F))) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(F))) \subset \mathrm{Cl}(f^{-1}(F))=f^{-1}(F).$$

This shows that $f^{-1}(F)$ is regular closed in X. For any $V \in RO(Y)$, $Y-V$ is regular closed in Y and hence $f^{-1}(V)=X-f^{-1}(Y-V) \in RO(X)$.

The following four corollaries are immediate consequences of Theorem 3.4 and the proofs are thus omitted. The first one is an improvement of Theorem A.

COROLLARY 3.5. *A weakly almost-open and w. c. function is δ-continuous.*

COROLLARY 3.6. *A w. c. function into an extremally disconnected space is δ-continuous.*

COROLLARY 3.7 ([1, Prop. 3.6]). *An open w. c. function is η-continuous.*

The following corollary answers the open question, Question B, posed by Rose [10].

COROLLARY 3.8. *An almost-open w. c. function is δ-continuous and hence almost-continuous.*

REMARK 3.9. It has been known that an open almost-continuous (in fact, δ-continuous) function is not necessarily continuous [5, Example 5.1].

4. Weakly-open functions

DEFINITION 4.1. A function $f: X \rightarrow Y$ is said to be *weakly-open* [10] if $f(U) \subset \mathrm{Int}(f(\mathrm{Cl}(U)))$ for every open set U of X.

The concepts of weak-openness and weak almost-openness are independent of each other, as the following examples show.

EXAMPLE 4.2. Let $X=\{a, b, c, d\}$ and

$$\sigma=\{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}.$$

Let $Y=\{x, y, z\}$ and $\tau=\{\phi, \{z\}, \{x, y\}, Y\}$. Define a function $f:(X, \sigma)\rightarrow (Y, \tau)$ as follows:

$$f(a)=f(c)=y, \ f(b)=x \text{ and } f(d)=z$$

Then, f is weakly almost-open, but it is not weakly-open.

EXAMPLE 4.3. Let $X=Y=\{a, b, c, d\}$, $\sigma=\{\phi, \{a, b, c\}, X\}$ and $\tau=\{\phi, \{c\}, \{a, d\}, \{a, c, d\}, Y\}$. Define a function $f:(X, \sigma)\rightarrow(Y, \tau)$ as follows:

$$f(a)=d, \ f(b)=a, \ f(c)=b \text{ and } f(d)=c.$$

Then, f is weakly-open, but it is not weakly almost-open.

THEOREM 4.4. *If a function $f:X\rightarrow Y$ is weakly-open and θ-continuous, then $f^{-1}(V)\in RO(X)$ for every $V\in RO(Y)$.*

Proof. First, we show that f is almost-continuous. Since f is θ-continuous, for each $x\in X$ and each open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(\mathrm{Cl}(U))\subset \mathrm{Cl}(V)$. Weak-openness of f implies $f(U)\subset \mathrm{Int}(f(\mathrm{Cl}(U)))\subset \mathrm{Int}(\mathrm{Cl}(V))$. Hence, f is almost-continuous. Let $V\in RO(Y)$. Then, $f^{-1}(V)$ is open in X [11, Theorem 2.2] and hence $f^{-1}(V)\subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V))$. Since f is almost-continuous, we have

$$\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))\subset \mathrm{Cl}(f^{-1}(V))\subset f^{-1}(\mathrm{Cl}(V)).$$

Moreover, f is weakly-open, and we obtain

$$f(\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))))\subset \mathrm{Int}(f(\mathrm{Cl}(f^{-1}(V))))\subset \mathrm{Int}(\mathrm{Cl}(V))=V.$$

Therefore, we have $\mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))\subset f^{-1}(V)$ and hence $\mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))=f^{-1}(V)$. This completes the proof.

COROLLARY 4.5. *A weakly-open θ-continuous function is δ-continuous and hence η-continuous.*

COROLLARY 4.6. *If $f:X\rightarrow Y$ is a weakly-open θ-continuous surjection and X is extremally disconnected, then Y is extremally disconnected.*

Proof. Let V be an open set of Y. Then, $\mathrm{Cl}(V)$ is regular closed in Y, and by Theorem 4.4 $f^{-1}(\mathrm{Cl}(V))$ is regular closed in X. Since X is extremally disconnected, $f^{-1}(\mathrm{Cl}(V))=\mathrm{Cl}(\mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ is open in X. The weak-openness of f implies $\mathrm{Cl}(V)\subset \mathrm{Int}(\mathrm{Cl}(V))$ and hence $\mathrm{Cl}(V)$ is open. This shows that Y is extremally disconnected.

A function $f:X\rightarrow Y$ is said to be *semi-open* [4] if $\mathrm{Int}(f(U))\neq\phi$ for each nonempty open set U of X. In [3], these functions are said to be *feebly-open*. It has been known that if a w.c. function $f:X\rightarrow Y$ satisfies these two conditions:

(a) $f^{-1}(V)\subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ for every $V\in RO(Y)$
and
(b) f is semi-open then f is η-continuous [1, Lemma 3.4].

THEOREM 4.7. *If a w.c. function $f:X\rightarrow Y$ satisfies the following two conditions:*
(a) $f^{-1}(V)\subset\mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ *for every* $V\in RO(Y)$ *and*
(b) f *is weakly-open,*
then $f^{-1}(V)\in RO(X)$ *for every* $V\in RO(Y)$.

Proof. By Theorem 4.4, it suffices to show that a *w.c.* function satisfying (a) is necessarily θ-continuous. Let $x\in X$ and V be any open neighborhood of $f(x)$. Since f is *w.c.*, $x\in f^{-1}(V)\subset\mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ [6, Theorem 1]. Put $U=\mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$, then U is an open neighborhood of x. Since $\mathrm{Int}(\mathrm{Cl}(V))\in RO(Y)$ and $\mathrm{Cl}(V)=\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))$, by Proposition 3.2 of [1] we obtain $f(\mathrm{Cl}(U))\subset\mathrm{Cl}(V)$. This shows that f is θ-continuous.

The following corollary shows that "*semi-open*" in Lemma 3.4 of [1] may be replaced by "*weakly-open*".

COROLLARY 4.8. *If a w.c. function $f:X\rightarrow Y$ satisfies these two conditions:*
(a) $f^{-1}(V)\subset\mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ *for every* $V\in RO(Y)$, *and*
(b) f *is weakly-open,*
then f is δ-continuous and hence η-continuous.

REMARK 4.9. The concepts of semi-openness and weak-openness are independent of each other even if the function is η-coninuous.

The function $f:(X,\sigma)\rightarrow(Y,\tau)$ in Example 2.2 is η-continuous and semi-open, but it is not weakly-open. Moreover, there exists a weakly-open continuous function which is not semi-open.

EXAMPLE 4.10. Let (Y,τ) be the space of Example 4.2. Let $X=\{a,b,c,d\}$ and $\sigma=\{\phi,\{a\},\{c\},\{a,b\},\{a,c\},\{c,d\},\{a,b,c\},\{a,c,d\},X\}$. Define a function $f:(X,\sigma)\rightarrow(Y,\tau)$ as follows:

$$f(a)=x,\ f(b)=y \text{ and } f(c)=f(d)=z.$$

Then, f is weakly-open and continuous, but it is not semi-open.

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