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RECURRENT HYPERSURFACES OF SYMMETRIC SPACES

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Introduction

A hypersurface N of a Riemannian manifold $\tilde{N}$ is said to be *quasiumbilical* if N has a principal curvature α with multiplicity $\geqslant \dim N - 1$. In particular when $\alpha=0$ or α is constant, then N is called a *hypercylinder* or a *quasihypersphere*. Bang-yen Chen and one of the authors studied the problem in the theory of submanifolds to find out the implications on the ambient space from the existence of a single submanifold of some particular type, and obtained the following [5].

THEOREM A. *Spheres, real projective spaces and their noncompact duals are the only irreducible symmetric spaces in which there exist hypercylinders.*

THEOREM B. *Spheres, real projective spaces, complex projective spaces and their noncompact duals are the only irreducible symmetric spaces in which there exist quasihyperspheres.*

So far the question of how many irreducible symmetric spaces $\tilde{N}$ contain an arbitrary quasiumbilical hypersurface N is not answered yet. With some additional intrinsic conditions on the quasiumbilical hypersurface N, namely the conditions to be *locally symmetric* or *conformally flat* or *Einsteinian*. it is known there are as few irreducible symmetric spaces $\tilde{N}$ as which admit hypercylinders [5]. In the present paper we extend one of these results to the following.

THEOREM. *Spheres, real projective spaces and their noncompact duals are the only irreducible symmetric spaces of dimension* $\geqslant 6$ *in which there exist recurrent quasiumbilical hypersurfaces.*

1. Preliminaries

In the following we will always understand $\tilde{N}$ to be an *irreducible symm-*

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etric space of dimension $\geqslant 6$. The metric tensor and the corresponding Levi-Civita connection of $\tilde{N}$ will be denoted by $\tilde{g}$ and $\tilde{\nabla}$. In particular $\tilde{N}$ is locally symmetric, that is its Riemann-Christoffel curvature tensor $\tilde{R}$ is parallel:

$$(1) \qquad \tilde{\nabla}\tilde{R}=0.$$

Let N be a hypersurface of $\tilde{N}$ and let ξ be a local unit normal vector field on N in $\tilde{N}$. Then the *formulas* of *Gauss and Weingarten* are given by

$$(2) \qquad \tilde{\nabla}_X Y=\nabla_X Y+h(X, Y)\xi,$$

$$(3) \qquad \tilde{\nabla}_X \xi=-AX,$$

whereby X and Y are arbitrary tangent vector fields on N, ∇ is the connection induced from $\tilde{\nabla}$ on N, h is the scalar valued *second fundamental form* of the hypersurface N of $\tilde{N}$ and A is the second fundamental tensor corresponding to ξ. A and h are related by

$$(4) \qquad h(X, Y)=g(AX, Y)$$

where g is the metric induced on N from the Riemannian metric $\tilde{g}$ of $\tilde{N}$, N is said to be *quasiumbilical* [1] [4] [6] if there exist two functions α and β and a unit 1-form ω on N such that

$$(5) \qquad h=\alpha g+\beta\omega\otimes\omega.$$

In particular when $\alpha=\beta=0$ N is a *totally geodesic hypersurface,* when $\beta=0$ N is an *umbilical hypersurface,* when $\alpha=0$ N is *hypercylinder* and when α is constant N is a *quasihypersphere* [5]. For a study of totally geodesic and totally umbilical submanifolds of symmetric spaces, see [2] [3]. Under the assumption of quasiumbilicity the *equations of Gauss and Codazzi* for the hypersurface N in $\tilde{N}$ are respectively

$$\begin{aligned}(6) \quad \tilde{R}(X, Y; Z, W)=&R(X, Y; Z, W)-\alpha^2[g(X, W)g(Y, Z)-g(X, Z)g(Y, W)]\\ &-\alpha\beta[g(W, X)\omega(Y)\omega(Z)+g(Y, Z)\omega(X)\omega(W)\\ &-g(X, Z)\omega(Y)\omega(W)-g(Y, W)\omega(X)\omega(Z)]\end{aligned}$$

and

$$\begin{aligned}(7) \quad \tilde{R}(X, Y; Z, \xi)=&(X\alpha)g(Y, Z)+(X\beta)\omega(Y)\omega(Z)+\beta(\nabla_X\omega)(Y)\omega(Z)\\ &+\beta\omega(Y)(\nabla_X\omega)(Z)-(Y\alpha)g(X, Z)-(Y\beta)\omega(X)\omega(Z)\\ &-\beta(\nabla_Y\omega)(X)\omega(Z)-\beta\omega(X)(\nabla_Y\omega)(Z),\end{aligned}$$

whereby X, Y, Z and W are arbitrary tangent vector fields on N and R is the Riemann-Christoffel curvature tensor of N.

In the following we assume that N is a *recurrent* Riemannian manifold [7], that is we assume the existence of a 1-form φ on N such that

$$(8) \qquad (\nabla_U R)(X, Y; Z, W)=\varphi(U)R(X, Y; Z, W)$$

for all X, Y, Z, W and U tangent to N.

2. Proof of Theorem

Based on the following result we must only consider the case $\beta \neq 0$.

LEMMA 1 [2]. *The only irreducible locally symmetric spaces which admit an umbilical hypersurface are real space forms.*

Consequently such a hypersurface is itself a real space form, that is a space of constant sectional curvature, and so in particular it is locally symmetric.
We now prove the following.

LEMMA 2. *Let N be a recurrent quasiumbilical hypersurface of a locally symmetric space. Then N is locally symmetric or N is a hypercylinder or a quasihypersphere.*

Proof. If $\varphi=0$ then N is locally symmetric and if $\alpha=0$ then N is a hypercylinder. We then prove this lemma by showing that when $\varphi\neq 0$ and $\alpha\neq 0$ the function α is constant. Therefore we derive the equation (6) of Gauss covariantly with respect to a vector U which is tangent to N; this yields:

$$
\begin{aligned}
(9)\quad \varphi(U)R(X,Y;Z,W) &= \alpha\beta\{g(X,W)[(\nabla_U\omega)(Y)\omega(Z)+\omega(Y)(\nabla_U\omega)(Z)]\\
&\quad -g(X,Z)[(\nabla_U\omega)(Y)\omega(W)+\omega(Y)(\nabla_U\omega)(W)]-g(Y,W)[(\nabla_U\omega)(X)\\
&\quad \omega(Z)+\omega(X)(\nabla_U\omega)(Z)]+g(Y,Z)[(\nabla_U\omega)(X)\omega(W)+\omega(X)(\nabla_U\omega)(W)]\\
&\quad +g(U,X)[(\nabla_W\omega)(Z)\omega(Y)+\omega(Z)(\nabla_W\omega)(Y)-(\nabla_Z\omega)(W)\omega(Y)\\
&\quad -\omega(W)(\nabla_Z\omega)(Y)]-g(U,Y)[(\nabla_W\omega)(Z)\omega(X)+\omega(Z)(\nabla_W\omega)(X)\\
&\quad -(\nabla_Z\omega)(W)\omega(X)-\omega(W)(\nabla_Z\omega)(X)]+g(U,Z)[(\nabla_Y\omega)(X)\omega(W)\\
&\quad +\omega(X)(\nabla_Y\omega)(W)-(\nabla_X\omega)(Y)\omega(W)-\omega(Y)(\nabla_X\omega)(W)]\\
&\quad -g(U,W)[(\nabla_Y\omega)(X)\omega(Z)+\omega(X)(\nabla_Y\omega)(Z)-(\nabla_X\omega)(Y)\omega(Z)\\
&\quad -\omega(Y)(\nabla_X\omega)(Z)]\}\\
&+\alpha\{(W\alpha)[g(U,X)g(Y,Z)-g(U,Y)g(X,Z)]+(Z\alpha)[g(U,Y)g(X,W)\\
&\quad -g(U,X)g(Y,W)]+(Y\alpha)[g(U,Z)g(X,W)-g(U,W)g(Y,Z)]\\
&\quad +(X\alpha)[g(U,W)g(Y,Z)-g(U,Z)g(Y,W)]+2(U\alpha)[g(X,W)g(Y,Z)\\
&\quad -g(X,Z)g(Y,W)]+[(W\beta)\omega(Z)-(Z\beta)\omega(W)][g(U,X)\omega(Y)\\
&\quad -g(U,Y)\omega(X)]+[(Y\beta)\omega(W)-(X\beta)\omega(Y)][g(U,Z)\omega(W)\\
&\quad -g(U,W)\omega(Z)]+(U\beta)[g(X,W)\omega(Y)\omega(Z)+g(Y,Z)\omega(X)\omega(W)\\
&\quad -g(X,Z)\omega(Y)\omega(W)-g(Y,W)\omega(X)\omega(Z)]\}\\
&+\beta\{(W\alpha)\omega(U)[g(Y,Z)\omega(X)-g(X,Z)\omega(Y)]+(Z\alpha)\omega(U)[g(X,W)\\
&\quad \omega(Y)-g(Y,W)\omega(X)]+(Y\alpha)\omega(U)[g(X,W)\omega(Z)-g(X,Z)\omega(W)]\\
&\quad +(X\alpha)\omega(U)[g(Y,Z)\omega(W)-g(Y,W)\omega(Z)]+(U\alpha)[g(X,W)\omega(Y)\\
&\quad \omega(Z)+g(Y,Z)\omega(X)\omega(W)-g(X,Z)\omega(Y)\omega(W)-g(Y,W)\omega(X)\omega(Z)]\}\\
&+\beta^2\{\omega(X)\omega(Z)[(\nabla_Y\omega)(W)+(\nabla_W\omega)(Y)]-\omega(X)\omega(W)[(\nabla_Y\omega)(Z)\\
&\quad +(\nabla_Z\omega)(Y)]+\omega(Y)\omega(W)[(\nabla_X\omega)(Z)+(\nabla_Z\omega)(X)]-\omega(Y)\omega(Z)\\
&\quad [(\nabla_X\omega)(W)+(\nabla_W\omega)(X)]\}.
\end{aligned}
$$

In the following we will consider the distributions $\mathcal{D}_\omega=\{X\in TN\,|\,\omega(X)=0\}$, $\mathcal{D}_\varphi=\{X\in TN\,|\,\varphi(X)=0\}$ and $\mathcal{D}=\mathcal{D}_\omega\cap\mathcal{D}_\varphi$. The dual vectors of 1-forms ω, $\varphi,\cdots$ will be denoted by $\bar{\omega},\bar{\varphi},\cdots$. If $\varphi\neq 0$ we may consider a unit vector $\bar{\Phi}$ such that

$$\bar{\varphi}=a\bar{\omega}+b\bar{\Phi},\quad g(\bar{\omega},\bar{\Phi})=0. \tag{10}$$

For X, Y, Z, W and U in $\mathcal{D}$, (9) becomes

$$\begin{aligned}(W\alpha)[g(U,X)g(Y,Z)-g(U,Y)g(X,Z)]+(Z\alpha)[g(U,Y)g(X,W)\\ -g(U,X)g(Y,W)]+(Y\alpha)[g(U,Z)g(X,W)-g(U,W)g(X,Z)]\\ +(X\alpha)[g(U,W)g(Y,Z)-g(U,Z)g(Y,W)]+2(U\alpha)[g(X,W)g(Y,Z)\\ -g(X,Z)g(Y,W)]=0.\end{aligned} \tag{11}$$

We can always choose $X=Z=U$ and $Y=W$ whereby X and Y are orthonormal vectors of $\mathcal{D}$. Then (11) implies that

$$V\alpha=0 \tag{12}$$

for all vectors V of $\mathcal{D}$. From (9) and (12) it follows that for $X=\bar{\omega}$ and $U, Y, Z, W\in\mathcal{D}$ we have

$$\begin{aligned}\beta\{-g(Y,W)(\nabla_U\omega)(Z)+g(Y,Z)(\nabla_U\omega)(W)+g(U,Y)[(\nabla_Z\omega)(W)\\ -(\nabla_W\omega)(Z)]+g(U,Z)(\nabla_Y\omega)(W)-g(U,W)(\nabla_Y W)(Z)\}\\ +(\bar{\omega}\alpha)[g(U,W)g(X,Z)-g(U,Z)g(Y,W)]=0.\end{aligned} \tag{13}$$

Thus for orthonormal vectors $Z=U$ and $Y=W$ in $\mathcal{D}$ we find that

$$\bar{\omega}\alpha=\beta[\nabla_Y\omega)(Y)-(\nabla_Z\omega)(Z)], \tag{14}$$

which implies that

$$\bar{\omega}\alpha=0 \tag{15}$$

and, if dim $N\geqslant 5$, that

$$(\nabla_A\omega)(A)=(\nabla_B\omega)(B)=\gamma \tag{16}$$

for all unit vectors A, B in $\mathcal{D}$. In a similar way we may obtain

$$\bar{\varphi}\alpha=0. \tag{16}$$

From (12), (15) and (16) we conclude that α is constant.

From [5] we know the following results.

LEMMA 3. *Real space forms are the only irreducible locally symmetric spaces which admit locally symmetric quasiumbilical hypersurfaces.*

LEMMA 4. *Real space forms are the only irreducible locally symmetric spaces which admit hypercylinders.*

LEMMA 5. *Real and complex space forms are the only irreducible locally symmetric spaces which admit quasihyperspheres.*

Here by complex space forms are meant the Kaehlerian manifolds of constant holomorphic sectional curvature. Concerning Lemma 4 we remark that hypercylinders of real space forms are themselves real space forms and thus in particular are locally symmetric.

We now prove the following property which, together with proofs of Lemma 1 - 5, is the main result of this paper.

LEMMA 6. *A complex space form of dimension* $\geqslant 6$ *does not admit a recurrent quasiumbilical hypersurface.*

Proof. From Lemma's 2, 3 and 4 it is sufficient to prove that the assumption of the existence of a recurrent quasihypersphere in a complex space form of dimension $\geqslant 6$ leads to a contradiction. So in particular we suppose that α is a nonzero constant. For U, Z in $\mathcal{D}$, $X=W=\bar{\varphi}$ and $Y=\bar{\omega}$, (9) becomes

$$(\nabla_U\omega)(Z)-g(U,Z)(\nabla_{\bar{\xi}}\omega)(\bar{\phi})=0. \tag{17}$$

Hence

$$(\nabla_{\bar{\xi}}\omega)(\bar{\phi})=\gamma \tag{18}$$

and

$$(\nabla_E\omega)(F)=0 \tag{19}$$

for all pairs E, F of perpendicular vectors in $\mathcal{D}$. Putting $X=\bar{\omega}$, $W=\bar{\phi}$ and choosing U, Y and Z in $\mathcal{D}$, (9) becomes

$$\begin{aligned} g(U,Y)[(\nabla_Z\omega)(\bar{\phi})-(\nabla_{\bar{\xi}}\omega)(Z)]+g(U,Z)(\nabla_Y\omega)(\bar{\phi}) \\ +g(Y,Z)(\nabla_U\omega)(\bar{\phi})=0. \end{aligned} \tag{20}$$

Thus for every vector V in $\mathcal{D}$ we have

$$(\nabla_V\omega)(\bar{\phi})=(\nabla_{\bar{\phi}}\omega)(V)=0. \tag{21}$$

From (16), (18), (19) and (21) we see that

$$(\nabla_X\omega)(Y)=(\nabla_Y\omega)(X) \tag{22}$$

for all vectors X, Y in $\mathcal{D}_\omega$. From Lemma 3.1 of [5] we know that (22) is a necessary and sufficient condition for the distribution $\mathcal{D}_\omega$ to be integrable, which then by the same arguments as given in the proof of Lemma 7.3 of [5] implies that α must vanish.

REMARK. From the above results we also have the following property. *Let N be a quasihypersphere of an irreducible locally symmetric space $\tilde{N}$ of dimension* $\geqslant 6$. *Then N is recurrent only when N is locally symmetric.*

Proof. Under the assumptions on N and $\tilde{N}$ the Theorem asserts that $\tilde{N}$ is a space of constant sectional curvature, say $\tilde{c}$. Thus

(23) $\tilde{R}(\tilde{X}, \tilde{Y}; \tilde{Z}, \tilde{W}) = \tilde{c}[\tilde{g}(\tilde{X}, \tilde{W})\tilde{g}(\tilde{Y}, \tilde{Z}) - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{g}(\tilde{Y}, \tilde{W})]$,

where $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and $\tilde{W}$ are arbitrary tangent vector fields on $\tilde{N}$. When $\alpha=0$ or $\beta=0$ then N is either a cylindrical or an umbilical hypersurface of a real space form and therefore it is itself a real space form. So in the following we have $\beta\neq 0$ and α is a constant different from 0. By (23) the equation (6) of Gauss becomes

(24) $R(X, Y; Z, W) = (\tilde{c}+\alpha^2)[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)]$
$+\alpha\beta[g(X, W)\omega(Y)\omega(Z) + g(Y, Z)\omega(X)\omega(W) - g(X, Z)\omega(Y)\omega(W)$
$-g(Y, W)\omega(X)\omega(Z)]$,

where X, Y, Z and W are tangent to N. Choosing $X, Y, Z, W\in\mathcal{D}$ and $U=\bar{\varphi}$, (9) gives

(25) $R(X, Y; Z, W) = 0.$

Together with (24) this implies that

(26) $\tilde{c} = -\alpha^2.$

From (16), (18), (19) and (21) we know that

(27) $(\nabla_X\omega)(Y) = g(X, Y)\gamma$

for all X, $Y\in\mathcal{D}_\omega$. Therefore, first putting $X=W=\bar{\omega}$, $Y=\bar{\varphi}$ and taking U and Z in $\mathcal{D}$ and next putting $X=U=\bar{\omega}$, $W=\bar{\Phi}$ and taking Y and Z in $\mathcal{D}$ in formula (9), we get

(28) $\bar{\Phi}\beta = \beta(\nabla_{\bar{\omega}}\omega)(\bar{\Phi})$

and

(29) $\alpha\beta g(Y, Z)(\nabla_{\bar{\omega}}\omega)(\bar{\Phi}) = \varphi(\bar{\omega})R(\bar{\omega}, Y; Z, \bar{\Phi})$,

such that

(30) $\alpha\beta(\bar{\Phi}\beta)g(Y, Z) = \varphi(\bar{\omega})R(\bar{\omega}, Y; Z, \bar{\Phi})$.

From (24), (26) and (30) we obtain

(31) $\bar{\Phi}\beta = 0.$

Then taking Y and Z in $\mathcal{D}$ and putting $U=\bar{\Phi}$ and $X=W=\bar{\omega}$, (9) gives

(32) $\varphi(\bar{\Phi})R(\bar{\omega}, Y; Z, \bar{\omega}) = 0.$

Thus by the equation (24) of Gauss we find that

(33) $\alpha\beta\varphi(\bar{\Phi})g(Y, Z) = 0$

which shows that

(34) $\varphi(\bar{\Phi}) = 0.$

From (10) it then follows that φ must be a scalar multiple of ω, say

(35) $\varphi = f\omega.$

When applied for $Y, Z\in\mathcal{D}$ and $X=\bar{\omega}$, the equation (7) of Codazzi becomes

(36) $\tilde{R}(\bar{\omega}, Y; Z, \xi) = -\beta(\nabla_Y\omega)(Z).$

By (23) this shows that

(37) $$(\nabla_Y \omega)(Z) = 0$$

for all $Y, Y \in \mathcal{D}$. Moreover, for $Y = Z = \bar{\omega}$ and $X \in \mathcal{D}$ the equation (7) of Codazzi becomes

(38) $$X\beta = \beta(\nabla_{\bar{\omega}} \omega)(X).$$

Choosing Y, Z and W in $\mathcal{D}$ and putting $X = U = \bar{\omega}$ in (9)) yields

(39) $$fR(\bar{\omega}, Y; Z, W) = \alpha\beta[g(Y, Z)(\nabla_{\bar{\omega}} \omega)(W) - g(Y, W)(\nabla_{\bar{\omega}} \omega)(Z)],$$

which by the equation (24) of Gauss gives

(40) $$f[g(Y, Z)(\nabla_{\bar{\omega}} \omega)(W) - g(Y, W)(\nabla_{\bar{\omega}} \omega)(Z)] = 0.$$

If f were non-zero, then for all X in $\mathcal{D}$ from (40) we obtain that

(41) $$(\nabla_{\bar{\omega}} \omega)(X) = 0,$$

and so from (38) we have

(42) $$X\beta = 0.$$

Then (37), (41) and the fact that ω is a unit 1-form imply that $\bar{\omega}$ is a parallel vector field on N. Therefore by definition

(43) $$R(X, \bar{\omega}; \bar{\omega}, W) = 0$$

for all X, W tangent to N which is in contradiction with (24). Thus $f = 0$ which combined with (35) shows that $\varphi = 0$.

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