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FIBERWISE PL INVOLUTIONS OF FIBERED 3-MANIFOLDS

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1. Introduction

As in our previous paper [4] with K. W. Kwun, an involution h of a 3-manifold X is said to be *fiberwise* if there is a fibering of X over S^1 such that each fiber is invariant under h. In this paper, we give a new characterization of fiberwise involutions of closed P^2-irreducible 3-manifolds. Unlike the situation in [4], however, involutions are not assumed to have fixed points.

A *fibering of a group* G (over $\mathbf{Z}$) is an epimorphism $\varepsilon_0 : G \to \mathbf{Z}$ whose kernel is finitely generated. Let X be a closed 3-manifold admitting a fibering over S^1 and consider the set of fiberings of $\pi_1(X, x_0)$, $x_0 \in X$. The automorphism group $\operatorname{Aut}\pi_1(X, x_0)$ of $\pi_1(X, x_0)$ acts from the right on this set by usual composition of maps. Moreover, since $\mathbf{Z}$ is abelian, this action of $\operatorname{Aut}\pi_1(X, x_0)$ induces an action of $\operatorname{Out}\pi_1(X)$, the group of outer automorphisms of $\pi_1(X, x_0)$. In order to deal with possibly free involutions h of X, identify $\pi_1(X, h(x_0))$ to $\pi_1(X, x_0)$ using a path between base points. Because this identification is unique up to inner automorphism, one can safely speak of the outer automorphism class, denoted by $\hat{h}_\#$ of $h_\#$. Then $\hat{h}_\#$ as an element of $\operatorname{Out}\pi_1(X)$ acts on the set of fiberings of $\pi_1(X, x_0)$.

THEOREM 1. *Let X be a closed P^2-irreducible 3-manifold, $x_0 \in X$. The following are equivalent for a PL involution h of X.*

1) *h is fiberwise,*

2) *$\pi_1(X, x_0)$ admits a fibering $\varepsilon_0 : \pi_1(X, x_0) \to \mathbf{Z}$ invariant under $\hat{h}^\#$ such that the covering space corresponding to* Ker ε_0 *admits an involution that covers h,*

3) *$\pi_1(X, x_0)$ admits a fibering $\varepsilon_0 : \pi_1(X, x_0) \to \mathbf{Z}$ invariant under $\hat{h}_\#$ such that* Ker ε_0 *contains an element of the form $[\gamma h \gamma]$ for some path γ from x_0 to $h(x_0)$.*

COROLLARY. *A PL involution h fixing a point x_0 of a closed P^2-irreducible 3-manifold X is fiberwise if and only if* the group *$\pi_1(X, x_0)$ admits a fibering $\varepsilon_0 : \pi_1(X, x_0) \to \mathbf{Z}$ such that $\varepsilon_0 \circ h_\# = \varepsilon_0$.*

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THEOREM 2. *Let h be a PL involution of a closed P^2-irreducible 3-manifold, $x_0 \in X$. If $\pi_1(X, x_0)$ admits a fibering $\varepsilon_0 : \pi_1(X, x_0) \to \mathbf{Z}$ invariant under $\hat{h}_\#$, then X admits a fibering $g : X \to S^1$ such that*

1) $\varepsilon_0 = g_\# : \pi_1(X, x_0) \to \mathbf{Z} = \pi_1(S^1)$,

2) $g \circ h = g$ *or* $g \circ h = a \circ g$, *where $a : S^1 \to S^1$ is the antipodal map.*

2. Covering homeomorphisms $\tilde{h}$

Throughout, let X, x_0 and h be as in Theorem 1 and suppose that the fibering $\varepsilon_0 : \pi_1(X, x_0) \to \mathbf{Z}$ is invariant under $\hat{h}_\#$.

An effect of identifying $\pi_1(X, x_1)$, $x_1 \in X$, to $\pi_1(X, x_0)$ is that the groups are fibered simultaneously. This simultaneous fibering can be visualized as follows. Factor the epimorphism ε_0 into the obvious composite

$$\pi_1(X, x_0) \to \pi_1(X, x_0)/\mathrm{Ker}\,\varepsilon_0 \xrightarrow{\tilde{\varepsilon}} \mathbf{Z}.$$

The group in the middle is canonically isomorphic to the group of covering transformations for the infinite cyclic regular covering $q : \tilde{X} \to X$ such that Ker $\varepsilon_0 = q_\# \pi_1(\tilde{X}, \tilde{x}_0)$, $x_0 \in q^{-1}(x_0)$. Thus, fibering $\pi_1(X, x_0)$ by ε_0 amounts to choosing a generator T of covering transformations for $q : \tilde{X} \to X$. This eliminates the role of x_0 and we have the fibering $\varepsilon_1 : \pi_1(X, x_1) \to \mathbf{Z}$ for any $x_1 \in X$ given by the composite:

$$\pi_1(X, x_1) \to \pi_1(X, x_1)/q_\# \pi_1(\tilde{X}, \tilde{x}_1) = \pi_1(X, x_0)/q_\# \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{\tilde{\varepsilon}} \mathbf{Z}.$$

Here, $\tilde{x}_1 \in q^{-1}(x_1)$ and the equality in the middle means that we are regarding both sides as covering transformations. In case $x_1 = h(x_0)$, observe that $\varepsilon_0 : \pi_1(X, x_0) \to \mathbf{Z}$ is invariant under $\hat{h}_\#$ precisely when $\varepsilon_0 = \varepsilon_1 \circ h_\#$.

In what follows, $q : \tilde{X} \to X$ and T are as in the above argument. The base point $\tilde{x}_0 \in q^{-1}(x_0)$ will be fixed once it is chosen. If γ is any path in X with origin x_0, $\tilde{\gamma}$ will denote its lift in $\tilde{X}$ with origin $\tilde{x}_0$.

Now let $\tilde{x}_1$ be any point in $q^{-1}(h(x_0))$. Because $\varepsilon_0 = \varepsilon_1 \circ h_\#$, $h_\# \circ q_\# \pi_1(\tilde{X}, \tilde{x}_0) = q_\# \pi_1(\tilde{X}, \tilde{x}_1)$. Then, there is a unique $\tilde{h} : \tilde{X} \to \tilde{X}$ with $\tilde{h}(\tilde{x}_0) = \tilde{x}_1$ such that $q \circ \tilde{h} = h \circ q$. Any map $\tilde{h} : \tilde{X} \to \tilde{X}$ with $q \circ \tilde{h} = h \circ q$ is a homeomorphism and is referred to as a covering hemeomorphism (for h).

LEMMA 1. *Each covering homeomorphism $\tilde{h}$ commutes with T.*

Proof. Let γ be a loop at x_0 such that $\tilde{\gamma}(1) = T(\tilde{x}_0)$. Both $[\gamma]$ and $h_\#[\gamma] = [h\gamma]$ represent the same covering transformation T because ε_0 is invariant under $\hat{h}_\#$. Since $\tilde{h}\tilde{\gamma}$ is the lift with origin $\tilde{h}(\tilde{x}_0)$ of $h\tilde{\gamma}$, $\tilde{h}r(1) = T(\tilde{h}\tilde{\gamma}(0))$. Hence $\tilde{h}T(\tilde{x}_0) = T\tilde{h}(\tilde{x}_0)$ and the lemma follows because of the unique lifting property.

LEMMA 2. *Either $\tilde{X}$ admits an involution covering h or $\tilde{h}\tilde{h} = T$ for some* *covering homeorphism* $\tilde{h}$.

Proof. If $\tilde{h}$ is a covering homeomorphism, $\tilde{h}\tilde{h}$ being a covering transformation, it is expressible as $\tilde{h}\tilde{h}=T^i$. All covering homeomorphisms are of the form $\tilde{h}T^k$ and $(\tilde{h}T^k)^2=T^{i+2k}$ by Lemma 1. If $i\neq 0$, 1, replace $\tilde{h}$ by $\tilde{h}T^k$ such that $i+2k=0$ or 1.

LEMMA 3. *$\tilde{X}$ admits an involution $\tilde{h}$ covering h if and only if there is a path γ in X joining x_0 to $h(x_0)$ such that* $[\gamma h\gamma]\in\operatorname{Ker}\varepsilon_0$.

Proof. If $\tilde{h}$ is a covering involution, choose any path $\tilde{\gamma}$ joining $\tilde{x}_0$ to $\tilde{h}(\tilde{x}_0)$ in $\tilde{X}$ and let $\gamma=q\tilde{\gamma}$. Then $\tilde{\gamma}\tilde{h}\tilde{\gamma}$ is a loop and $\gamma h\gamma\in q_\#\pi_1(\tilde{X},\tilde{x}_0)=\operatorname{Ker}\varepsilon_0$.
Conversely, if there is a path γ from x_0 to $h(x_0)$ such that $[\gamma h\gamma]\in\operatorname{Ker}\varepsilon_0$, let $\tilde{h}$ be the covering homeomorphism such that $\tilde{h}(\tilde{x}_0)=\tilde{\gamma}(1)$. Because $q(\tilde{\gamma}\tilde{h}\tilde{\gamma})=\gamma h\gamma$ represents an element of Ker $\varepsilon_0=q_\#\pi_1(\tilde{X},\tilde{x}_0)$, the path $\tilde{\gamma}\tilde{h}\tilde{\gamma}$ in fact is a loop at $\tilde{x}_0$. Hence $\tilde{h}\tilde{h}(\tilde{x}_0)=\tilde{h}\tilde{\gamma}(1)=\tilde{\gamma}\tilde{h}\tilde{\gamma}\ (1)=\tilde{x}_0$. This shows that $\tilde{h}$ is an involution as $\tilde{h}\tilde{h}$ must be a covering transformation.

Let F and $\tilde{F}$ be the fixed point sets of h and $\tilde{h}$ respectively.

LEMMA 4. *If h is an involution, then* $\tilde{F}=q^{-1}(F)$

Proof. The inclusion $\tilde{F}\subset q^{-1}(F)$ is obvious. Before proving the reverse inclusion, observe that the outer automomorphism class $\hat{h}_\#$ does not depend on the chosen base point x_0 and the effect of $\hat{h}_\#$ on the simultaneous fiberings $\varepsilon_1:\pi_1(X,x_1)\to\mathbf{Z}$ for various $x_1\in X$ is compatible with isomorphisms of fundamental groups induced by paths joining base points. This is clear because of the commutative square

$$\begin{array}{ccc}\pi_1(X,x_0) & \xrightarrow{\sigma_r} & \pi_1(X,x_1)\\ \big\downarrow h_\# & & \big\downarrow h_\# \\ \pi_1(X,h(x_0)) & \xrightarrow{\sigma_{hr}} & \pi_1(X,h(x_1))\end{array}$$

where γ is any path joining x_0 to x_1 and σ_r, σ_{hr} are induced by γ, $h\gamma$. Thus the action of $\hat{h}_\#$ is a base point free notion and, under our hypothesis, $\hat{h}_\#$ leaves all ε_1 invariant.
Now let $\tilde{x}_1$ be any point with $x_1=q(\tilde{x}_1)$ in F Because $\varepsilon_1:\pi_1(X,x_1)\to\mathbf{Z}$ is invariant under $\hat{h}_\#$, there is a covering homeomophism $\tilde{h}_1:\tilde{X}\to\tilde{X}$ such that $\tilde{h}_1(\tilde{x}_1)=\tilde{x}_1$. Hence $\tilde{h}_1=\tilde{h}T^k$. But $\tilde{h}_1$ is an involution because it fixes $\tilde{x}_1$ and we have $T^{2k}=(\tilde{h}T^k)^2=\tilde{h}_1{}^2=T^0$. Hence $\tilde{h}_1=\tilde{h}$ and $\tilde{h}(\tilde{x}_1)=\tilde{x}_1$. This completes the proof of Lemma 4.

3. Proof of Theorems 1 and 2

The implication (1) $\Rightarrow$ (3) in Theorem 1 is obvious, while (3) $\Rightarrow$ (2)

follows from Lemma 3. For (2) $\Rightarrow$ (1), regard $\tilde{X}$ as a product $M \times R^1$ where M is a closed surface with $\pi_1(M) \approx \mathrm{Ker}\, \varepsilon_0$. As in [4,] the covering involution $\tilde{h}$ is equivalent to $\alpha \times 1_{R^1}$ for suitable involution α of M. Actually, the proof of [4, Lemma 2.3] shows in the present case that $\tilde{h}$ is equivalent with $\alpha \times \lambda$, where λ is an involution of R^1. Because $\varepsilon_0 : \pi_1 (X, x_0) \to \mathbf{Z}$ is invariant under $\hat{h}_\#$, λ must be orientation preserving. Hence $\lambda = 1_{R^1}$. After this point, the proof is the same as in [4]. First fiber the orbit space of h using the fact that T also acts on the orbit space of $\tilde{h}$ and then obtain the desired fibering of X making h fiberwise. The argument of [4] works word to word in the present situation.

We now pass to Theorem 2. The conclusion follows form Theorem 1 if there is a covering involution on $\tilde{X}$. Otherwise, let $\tilde{h}$ be the covering homeomorphism of Lemma 2 such that $\tilde{h}\tilde{h} = T$. The orbit space Y of h is then identical with the orbit space of the free $\mathbf{Z}$-action on $\tilde{X}$ generated by $\tilde{h}$. Because $\tilde{h}\tilde{h} = T$ has no fixed point, $\tilde{h}$ acts freely indeed. Deneote by $\bar{q} : \tilde{X} \to Y$ the orbit map of this $\mathbf{Z}$-action given by $\tilde{h}$. We have the exact sequence

$$1 \to \bar{q}_\# \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(Y, y_0) \xrightarrow{\bar{\varepsilon}} \mathbf{Z} \to 0$$

where $\mathbf{Z}$ is generated by $\tilde{h}$, $y_0 = \bar{q}(\tilde{x}_0)$. Then $\mathrm{Ker}\ \bar{\varepsilon} \approx \pi_1(\tilde{X}, \tilde{x}_0) \approx \mathrm{Ker}\ \varepsilon_0$ is finitely generated; Ker $\bar{\varepsilon}$ is not isomorphic to $\mathbf{Z}_2$ or 0 either, because X is P^2-irreducible. Hence, by Stallings [6] there is a fibering $\bar{g} : Y \to S^1$ such that

$$\bar{\varepsilon} = \bar{g}_\# : \pi_1(Y, y_0) \to \mathbf{Z} = \pi_1(S^1, s_0).$$

Let M_0 be a component of $(\bar{g}\bar{q})^{-1}(s_0)$, let W be the part of $\tilde{X}$ bounded by M_0 and $\tilde{h}(M_0)$, and let $W' = W \cup \tilde{h}(W)$. Y can be obtained from W by identifying each $\tilde{x} \in M_0$ to $\tilde{h}(\tilde{x})$, and X is obtained from W' by identifying M_0 to $T(M_0)$ using $T = \tilde{h}\tilde{h}$. Regarding S^1 to be $[0, 1]$ with end points identified, we obtain the map $p : W \to [0, 1]$ such that $\bar{g}$ equals the composite of p with the obvious map $[0, 1] \to S^1$. Let $p' : W' \to [0, 2]$ be defined by $p' = p$ on W and $p'(\tilde{x}) = 1 + p(\tilde{h}^{-1}(\tilde{x}))$ for $\tilde{x} \in \tilde{h}(W)$. Regarding S^1 as $[0, 2]$ with end points identified one sees that p' gives a fibering $g : X \to S^1$. This g satisfies conditions required in Theorem 2.

4. Examples

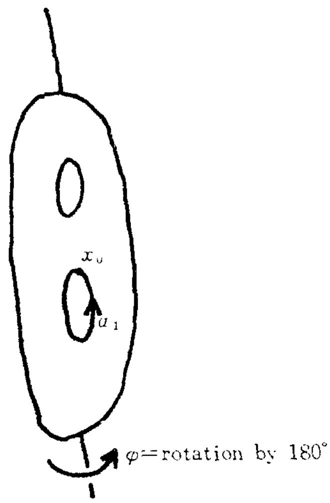
Theorem 1 can be used to recognize fiberwise involutions in certain practical situations. In the following, we give two examples along this line.

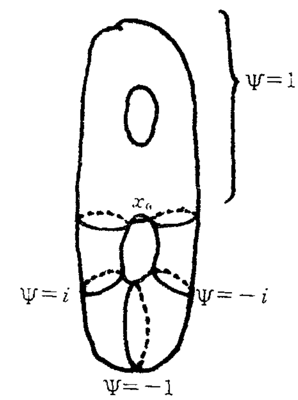
EXAMPLE 1. If M is an orientable closed surface of genus $g > 1$, $M \times S^1$ admits a fiberwise involution h different from product involution even though the fixed point set of h consists of circles of the form $x \times S^1$. In fact, the example of [4, §7] enjoys this property. The involution h is given by

$h(x, s)=(\varphi(x)\psi(x)s)$ where φ is the involution with $2g+2$ fixed points as in fig. 1, and $\psi: M \to S^1$ is a surjective map with values indicated in fig. 2 such that ψ(x) and $\psi(\varphi(x))$ are complex conjugates each other. In the presentation

$$\pi=\langle a_1, b_1, \ldots, a_g, b_g, t;\ [a_i, t]=[b_i, t]=1, \Pi[a_i, b_i]=1\rangle$$

of $\pi_1(M\times S^1, (x_0, s_0))$, generators can be so chosen that all a_i and b_i except





a_1 can be represented by loops in the part of $M=M\times S_0$ with $\psi=1$, a_1 is represented by the arrowed circle in fig. 1, and that t is represented by $x_0\times S^1$ suitably oriented. Define $\varepsilon_0: \pi \to \mathbf{Z}$ by $\varepsilon_0(a_1)=1$, $\varepsilon_0(a_i)=0$ for $i>1$, $\varepsilon_0(b_i)=0$ for all i, and $\varepsilon_0(t)=2$. By [1, p. 114], ε_0 is a well-defined fibering such that Ker ε_0 is isomorphic to the fundamental group of the orientable closed surface of genus $2(g-1)+1>g$. Observe that $h_\#(a_i)$ and a_i^{-1} belong to the same homology class for $i>1$. Similary for $h_\#(b_i)$ and b_i^{-1} for each i. Since $h_\#(a_1)=a_1^{-1}t$ and $h_\#(t)=t$, we have $\varepsilon_0\circ h_\#=\varepsilon_0$. The conclusion follows from Theorem 1.

EXAMPLE 2. Certain involutions can be fiberwise in infinitely many distinct ways. Let M be as in Example 1 and let h be any PL involution of $M\times S^1$ isotopic to $1_{M\times S^1}$. We show that $M\times S^1$ admits infinitely many inequivalent fiberings making h fiberwise. To this end, assume by [5] that $h=1_M\times a$, a being the antipodal map of S^1. Then the injective surfaces F of genera $k(g-1)+1$ in [3, Corollary 4. 7] can be assumed invariant under h for all even numbers $k\geq 2$. This situation can be better visualized by looking at figures in [2, Examples III. 14]. Again by [3], these surfaces serve as fibers for suitable fiberings of $M\times S^1$. Since $h_\#$ belongs to the trivial outer automorphism class, the assertion can be readily seen by looking at the fiberings of the fundamental group induced by such geometric fiberings with h-invariant fibers.

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