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ON THE STRUCTURE OF THE HALL-YAMADA SEMIGROUPS

BY DONG KIE KIM

1. Introduction

An *orthodox* semigroup is defined as a regular semigroup in which the idempotents form a subsemigroup. The class of orthodox semigroups thus includes both the class of inverse semigroups and the class of bands. Fantham [6], Yamada [15] and Petrich [12] have studied the case where the semigroup is also a union of groups. Specializing in another direction, Yamada [16] have studied the case where the band of idempotents of the semigroup is normal. Recentiy, the structure of orthodox semigroups in general has been clarified by Yamada [17] and Hall [8] independently. More recently, Hall [9] has generalized the Munn semigroup further in the case of a general regular semigroup.

Let S be an orthodox semigroup with band B of idempotents. The Hall semigroup W_B of the band B plays an important role in the structure theory to be discussed in this paper. Many of the idea involved are from Yamada's paper [19].

Our main theorem in this paper is Theorem 3. 3. This theorem asserts that the Hall-Yamada semigroup $S=\mathscr{H}(B, T, \phi)$ is an orthodox semigroup whose band of idempotents is isomorphic to B and that if γ is the minimum inverse semigroup congruence on S then $S/\gamma\cong T$. Conversely, if S is an orthodox semigroup whose band of idempotents is B then there is an idempotent-separating homomorphism $\theta : S/\gamma\to W_B/\gamma_1$ whose range contains all the idempotents of W_B/γ_1 and such that $S\cong\mathscr{H}(B, S/\gamma, \theta)$, where γ_1 is the minimum inverse semigroup congruence on the Hall semigroup W_B of B.

In section 2 we discuss basic properties of semigroups which are essential to understand our main theorem.

The notation and the terminalogy in this paper are standard. They are taken from [4]. Let ρ be a congruence on a semigroup S. Then S/ρ denotes the factor semigroup of S modulo ρ, and $\rho^{\natural} : S\to S/\rho$ denotes the natural homomorphism of S onto S/ρ. Let X be a set. Then $\mathscr{T}(X)$ means the semigroup of all transformations of X, and $\mathscr{PT}(X)$ means the semigroup

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of all partial transformations of X. The group of all permutations of X is denoted by $\mathcal{G}(X)$. By the (left-right) *dual* S^* of a semigroup S we mean the semigroup $(S, \circ)$, the elements of which are the same as those of S, and in which the binary operation $\circ$ is defined by $a\circ b=ba$ for all a, b in S.

2. Preliminaries

In this section we shall state several propositions which are useful in the next section. The proofs of propositions shall be omitted.

An element a of a semigroup S is called *regular* if $a\in aSa$. A semigroup S is called *regular* if every element of S is regular. Two elements a and b of a semigroup S are said to be *inverses* of each other if $aba=a$ and $bab=b$. By an *inverse semigroup* we mean a semigroup in which every element has a unique inverse. A *band* is a semigroup in which every element is idempotent.

DEFITION 2. 1. For each element a of a semigroup S, let

$$V(a)=\{b\in S : b \text{ is an inverse of } a\}$$

It is easy prove the following proposition.

PROPOSITION 2. 2 *Let S be an orthodox semigrovp. Then the relation γ on S defined by*

$$\gamma=\{(x, y) \in S\times S : V(x)=V(y)\}$$

is a congruence on S.

Moreover, it is the smallest inverse semigroup congruence on S,

DEFINITION 2. 3. Let S be a semigroup. Define relations $\mathcal{L}$ and $\mathcal{R}$ on S by

$$\mathcal{L}=\{(a, b)\in S\times S : a\cup Sa=b\cup Sb\},$$
$$\mathcal{R}=\{(a, b)\in S\times S : a\cup aS=b\cup bS\}.$$

Then $\mathcal{L}$ and $\mathcal{R}$ are a right and left congruence, respectively.

The relations $\mathcal{L}$ and $\mathcal{R}$ commute and so the relation $\mathcal{D}=\mathcal{L}\circ\mathcal{R}=\mathcal{R}\circ\mathcal{L}$ is the smallest equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$. Moreover, the relation $\mathcal{H}=\mathcal{L}\cap\mathcal{R}$ is an equivalence relation.

We denote the $\mathcal{L}$-class, the $\mathcal{R}$-class, the $\mathcal{D}$-class and the $\mathcal{H}$-class clontaining an element a by L_a, R_a, D_a and H_a, respectively.

It is known that a congruence ρ on a regular semigroup S is idempotent-separating if and only if $\rho\subseteq\mathcal{H}$. In particular, the congruence $\mathcal{H}^\#$, the largest congruence contained in $\mathcal{H}$, is the maximum idempotent-separating congruence on a regular semigroup S. Moreover, the following proposition holds.

PROPOSITION 2.4. *Let S be an inverse semigroup with semilattice E of idempotents. Then the relation*

$$\mu=\{(a,b)\in S\times S : a^{-1}ea=b^{-1}eb \text{ for all } e\in E\}$$

is the maximum idempotent-separating congruence on S.

Let S be an orthodox semigroup with semilattice B of idempotents. And let γ be the congruence on S defined in Proposition 2.2 and let

$$\varepsilon=\gamma\cup(B\times B).$$

Then ε is a congruence on B, and there exists a monomorphism $\eta : B/\varepsilon\rightarrow S/r$ which commutes the following diagram.

$$\begin{array}{ccc} S/r & \xrightarrow{\eta} & B/\varepsilon \\ \gamma^{\natural}\uparrow & & \uparrow\varepsilon^{\natural} \\ S & \xrightarrow{inc} & B \end{array}$$

Therefore, the semilattice of idempotents of the maximum inverse semigroup homomorphic image of an orthodox semigroup S is isomorphic to the maximum semilattice homomorphic image of the band B of idempotents of S. Furthermore, the following holds.

PROPOSITION 2.5. *Let S be an orthodox semigroup with band B of idempotents. If $\mu=\mathcal{H}^{\#}$ is the maximum idempotent-separating congruence on S, then $(a,b)\in\mu$ if and only if there exist $a'\in V(a)$ and $b'\in V(b)$ such that for any $x\in B$*

$$a'xa=b'xb \quad and \quad axa'=bxb'$$

By a *representation* of a semigroup S by partial transformations of a set X we mean a homomorphism $\varphi : S\rightarrow\mathcal{PT}(X)$ of S into $\mathcal{PT}(X)$, where $\mathcal{PT}(X)$ is the semigroup of all partial transformations of X. It is known that a mapping $\varphi : S\rightarrow\mathcal{PT}(S)$ which associates with each a of S an element δ_a defined by

$$\delta_a=\{(x,y)\in S\times S : y=xa \text{ and } (x,y)\in\mathcal{R}\}$$

is a representation of a semigroup S by partial transformations. But this representation is not in general faithful. In the case of a regular semigroup we can overcome this disadvantage by simultaneously considering the (left-right) dual of δ_a. Let $\mathcal{PT}^*(S)$ denote the dual semigroup of $\mathcal{PT}(S)$. We have the following result.

PROPOSITION 2.6. *Let S be a regular semigroup. For each a in S define*

$$\delta_a=\{(x,y)\in S\times S : y=xa \text{ and } (x,y)\in\mathcal{R}\},$$

$$\gamma_a=\{(x,y)\in S\times S : y=ax \text{ and } (x,y)\in\mathcal{L}\}.$$

Then the representation $\alpha : S\rightarrow\mathcal{PT}(S)\times\mathcal{PT}^*(S)$ defined by $a\alpha=(\delta_a,\gamma_a)$ is faithful.

Let S be an orthodox semigroup with band B of idempotents. Then we can define, for each a in S, a mapping $\rho_a : B/\mathcal{L} \to B/\mathcal{L}$ by

$$L_x\rho_a = L_{a'xa},$$

where a' is an arbitrarily chosen inverse of a. In particular, if $e \in B$ then we have $L_x\rho_e = L_{exe}$.

Note that if S is an inverse semigroup (so that B is a semilattice) then $L_x = \{x\}$ and $L_{a'xa} = \{a^{-1}xa\}$. By dual arguments we can define, for each a in S, a mapping $\lambda_a : B/\mathcal{R} \to B/\mathcal{R}$ by

$$R_x\lambda_a = R_{axa'},$$

where a' is an arbitrarily chosen inverse of a.

Now we have the following result.

PROPOSITION 2.7. *Let S be an orthodox semigroup with band B of idempotents. Let $\xi : S \to \mathcal{T}(B/\mathcal{L})\mathcal{T}^*(B/\mathcal{R})$ be a mapping defined by*

$$a\xi = (\rho_a, \lambda_a),$$

where ρ_a and λ_a are given by

$$L_x\rho_a = L_{a'xa}, \quad R_x\lambda_a = R_{axa'} \quad (x \in B).$$

Then ξ is a homomorphism whose kernel is the maximum idempotentseparating congruence μ on S.

Let B be a band and E a semilattice of B. For each e in B define $Ee = \{x \in E : x \leqslant e\}$. Then it is easy to see that $eBe = Ee$. We denote eBe by $\langle e \rangle$. Note that for all x, y in B

$$\langle x \rangle = \langle y \rangle \Leftrightarrow x = y.$$

We define

$$\mathcal{U} = \{(e, f) \in B \times B : \langle e \rangle \cong \langle f \rangle\}$$

and write $W_{e,f}$ for the set of all isomorphims of $\langle e \rangle$ onto $\langle f \rangle$. Note that if $g \in \langle e \rangle$ and $\alpha \in W_{e,f}$ then

$$\langle g \rangle \alpha = \langle g\alpha \rangle \quad \text{and} \quad e\alpha = f.$$

If $(e, f) \in \mathcal{U}$ and $\alpha \in W_{e,f}$, we may define $\alpha_l \in \mathcal{G}(B/\mathcal{L})$ and $\alpha_r \in \mathcal{G}(B/\mathcal{R})$ by

$$L_x\alpha_l = L_{x\alpha}, \quad R_x\alpha_r = R_{x\alpha} \quad (x \in \langle e \rangle).$$

Now let S be an orthodox semigroup with band B of idempotents. Let $a \in S$ and $a' \in V(a)$. Denoting aa' by e and $a'a$ by f, we observe that the mapping $\rho_a \in \mathcal{T}(B/\mathcal{L})$ defined in Proposition 2.7 may be expressed as $\rho_e\theta_l$, where θ is an element of $W_{e,f}$ which is the mapping given by

$$x\theta = a'xa \qquad (x \in \langle e \rangle).$$

Similarly, the mapping $\lambda_a \in \mathcal{T}(B/\mathcal{R})$ defined in Proposition 2.7 may be expressed as $\lambda_f\theta_r^{-1}$. The range of the mapping $\xi : S \to \mathcal{T}(B/\mathcal{L}) \times \mathcal{T}^*(B/\mathcal{R})$ defined by $a\xi = (\rho_a, \lambda_a)$ is thus contained in the subset

$$W_B=\{(\rho_e\alpha_l, \lambda_f\alpha_r^{-1}) : \alpha\in W_{e,f},\ (e,f)\in\mathcal{U}\}$$

of $\mathscr{T}(B/\mathscr{L})\times\mathscr{T}^*(B/\mathscr{R})$. We say that W_B is the *Hall semigroup* of the band B. Now we have the following result.

PROPOSITION 2. 8. *Let B be a band and let*

$$W_B=\{(\rho_e\lambda_l,\ \lambda_f\alpha_r^{-1}) : \alpha\in W_{e,f},\ (e,f)\in\mathcal{U}\}.$$

Then

(1) *W_B is a subsemigroup of $\mathscr{T}(B/\mathscr{L})\times\mathscr{T}^*(B/\mathscr{R})$.*

(2) *W_B is orthodox, with band of idempotents $B^*=\{(\rho,\lambda_e) : e\in B\}$ isomorphic to B.*

(3) *If B^* is identified with B, then, in W_B,*

$$\mathscr{D}\cup(B\times B)=\mathcal{U}.$$

3. Main Theorem

Let S be an orthodox semigroup with band B of idempotents. Then, by Proposition 2. 7, the mapping $\xi : a\rightarrow(\rho_a, \lambda_a)$ of S onto W_B is not in general one-one. Indeed its kernel is μ. However, since we have

$$\gamma\cap\mu\subseteq\gamma\cap\mathscr{H}=I_S,$$

the homomorphism $\eta : S\rightarrow W_B\times S/\gamma$ defined by

$$a\eta=((\rho_a, \lambda_a), a\gamma) \qquad (1)$$

is one-one (see Proposition 2. 2.),

If γ_1 is the minimum inverse semigroup congruence on W_B, then $\xi\gamma_1^\natural$ is a homomorphism of S into the inverse semigroup W_B/γ_1 which must factor through S/γ in accordance with the commutative diagram

$$\begin{array}{ccc} S/\gamma & \xrightarrow{\theta} & W_B/\gamma_1 \\ \gamma^\natural\uparrow & & \uparrow\gamma_1^\natural \\ S & \xrightarrow[\xi]{} & W_B \end{array} \qquad (2)$$

The homomorphism θ is uniquely determined, and we have the following lemma.

LEMMA 3. 1. *The homomorphism θ is idempotent-separating. The range of θ contains all the idempotents of W_B/γ_1.*

Proof. Let $e\gamma$ and $f\gamma$ be idempotents in S/γ (where $e,f\in B$) and suppose that $(e\gamma)\theta=(f\gamma)\theta$. Then $e\xi\gamma_1=f\xi\gamma_1$, that is, $(e\xi, f\xi)\in\gamma_1\cap(B^*\times B^*)$, where B^* is the band of idempotents of W_B. It follows that the idempotents $e\xi, f\xi$ in W_B are $\mathscr{G}$-equivalent in B^*. Since $\xi|B$ is an isomorphism of B onto the band B^*, it follows that e and f are $\mathscr{G}$-equivalent in B. Hence we have $e\gamma=f\gamma$.

Any idempotent in W_B/γ_1 is expressible as $(\rho_e, \lambda_e)\gamma_1$, where (ρ_e, λ_e) is an idempotent in W_B. Thus it is expressible as $e\xi\gamma_1$ for some idempotent e in S. The commutativity of diagram (2) then enables us to express our idempotent as $(e\gamma)\theta$. Hence every idempotent in W_B/γ_1 lies in the range of θ.

Any element $((\rho_a, \lambda_a), a\gamma)$ in the range of η has the property that

$$(\rho_a, \lambda_a)\gamma_1 = a\xi\gamma_1 = a\gamma\theta = (a\gamma)\theta.$$

Conversely, we shall show that if $(x, a\gamma) \in W_B \times S/\gamma$ is an element such that $x\gamma_1 = (a\gamma)\theta$ then $(x, a\gamma) = b\eta$ for some b in S. In other words, we establish

PROPOSITION 3.2. *Let S be an orthodox semigroup with band B of idempotents. The mapping $\eta : S \to W_B \times S/\gamma$ defined by* (1) *is an isomorphism of S onto*

$$\{(x, a\gamma) \in W_B \times S/\gamma : x\gamma_1{}^{\natural} = (a\gamma)\theta\},$$

the *spined product* of W_B and S/γ with respect to W_B/γ_1, $\gamma_1{}^{\natural}$ and θ.

Proof. It remains to show that η is onto. Let $(x, a\gamma) \in W_B \times S/\gamma$ such that $x\gamma_1 = (a\gamma)\theta$. Then $x\gamma_1 = (a\gamma)\theta = (\rho_a, \lambda_a)\gamma_1$ so that $V(x) = V(\rho_a, \lambda_a)$ in W_B. Now for any inverse c of a in S it is easy to verify that $(\rho_c, \lambda_c) \in V(\rho_a, \lambda_a)$ in W_B. Hence $(\rho_c, \lambda_c) \in V(x)$ and so both $(\rho_c, \lambda_c)x$ and $x(\rho_c, \lambda_c)$ are idempotents in W_B. Therefore, there exist e, f in B such that

$$(\rho_c, \lambda_c)x = (\rho_e, \lambda_e), \quad x(\rho_c, \lambda_c) = (\rho_f, \lambda_f).$$

As a consequence we have that

$$(\rho_e, \lambda_e)\ \mathcal{R}\ (\rho_c, \lambda_c), \quad (\rho_c, \lambda_c)\ \mathcal{L}\ (\rho_f, \lambda_f)$$

in W_B. That is, $e\xi\,\mathcal{R}\,c\xi$ and $c\xi\,\mathcal{L}\,f\xi$ in W_B. Examining the first of these, we deduce that $e\xi$ and $c\xi$ are $\mathcal{R}$-equivalent in $S\xi$. Thus there exist u, v in S such that

$$e\xi = (c\xi)(u\xi), \quad c\xi = (e\xi)(v\xi),$$

that is, such that

$$(e, cu) \in \xi\circ\xi^{-1}, \quad (c, ev) \in \xi\circ\xi)^{-1}$$

Now $\xi\circ\xi^{-1} = \mu \subseteq \mathcal{H} \subseteq \mathcal{R}$ and so there exist x and y in S such that

$$e = cux, \quad c = evy.$$

We conclude that $e\mathcal{R}c$ in S. Similarly, $c\mathcal{L}f$ in S.

Now it assures us that the $\mathcal{H}$-class $L_e \cap R_e$ contains an inverse b of c. It follows that in W_B the element (ρ_b, λ_b) is an inverse of (ρ_c, λ_c) and that it is $\mathcal{L}$-equivalent to (ρ_e, λ_e) and $\mathcal{R}$-equivalent to (ρ_f, λ_f). Since x also has these properties we conclude that $x = (\rho_b, \lambda_b)$. Note that $b\gamma$ and $a\gamma$ are both inverses of $c\gamma$ in the inverse semigroup S/γ. Hence $b\gamma = a\gamma$ and so

$$(x, a\gamma) = ((\rho_b, \lambda_b), b\gamma) = b\eta.$$

It is natural to make the following construction. Let B be a band and let T be an inverse semigroup whose semilattice of idempotents is isomorphic to B/ε. Let γ_1 be the minimum inverse semigroup congruence on the Hall semigroup W_B of B. Then W_B/γ_1 is an inverse semigroup whose semilattice of idempotents is isomorphic to B/ε. Let $\Psi : T \to W_B/\gamma_1$ be an idempotent-separating homomorphism whose range contains all the idempotents of W_B/γ_1. Then we denote the spined product

$$S = \{(x, t) \in W_B \times T : x\gamma_1{}^{\natural} = t\Psi\} \tag{3}$$

of W_B and T with respect to W_B/γ_1, $\gamma_1{}^{\natural}$ and Ψ by $\mathscr{H}(B, T, \Psi)$. This semigroup $\mathscr{H}(B, T, \Psi)$ is called the *Hall-Yamada semigroup* determined by the band B, the inverse semigroup T and the idempotent-separating homomorphism Ψ.

THEOREM 3.3. *Let B, T, γ_1 and Ψ be as above. Then the Hall-Yamada semigroup $S=\mathscr{H}(B, T, \Psi)$ is an orthodox semigroup whose band of idempotents is isomorphic to B. If γ is the minimum inverse semigroup congruence on S, then $S/\gamma \cong T$.*

Conversely, if S is an orthodox semigroup whose band of idempotents is B, then there exists an idempotent-separating homomorphism $\theta : S/\gamma \to W_B/\gamma_1$ whose range contains all the idempotents of W_B/γ_1 and such that $S \cong \mathscr{H}(B, S/\gamma, \theta)$.

Proof. Note that the second half of this theorem is a restatement of Lemma 3.1 and Proposition 3.2.

To prove the first half we shall show that $S=\mathscr{H}(B, T, \Psi)$ is regular. It is obvious that $W_B \times T$ is regular; indeed we can say that the set of inverses of an element (x, t) in $W_B \times T$ is $V(x) \times \{t^{-1}\}$. If the element (x, t) is in S, that is, if $t\Psi = t_1{}^{\natural}$, then for every x' in $V(x)$ the elements $x'\gamma_1{}^{\natural}$ and $t^{-1}\Psi$ are both inverses of the element $x\gamma_1{}^{\natural} = t\Psi$ of the inverse semigroup W_B/γ_1; hence $x'\gamma_1 = t^{-1}\Psi$. and so $(x', t^{-1}) \in S$. Thus S is a regular subsemigroup of $W_B \times T$, and we have shown moreover that the set of inverses of an element (x, t) of S is $V(x) \times \{t^{-1}\}$.

That S is orthodox follows immediately from the fact that W_B and T are orthodox and from the fact that an element (x, t) of $W_B \times T$ is idempotent if and only if x is an idempotent of W_B and t is an idempotent of T.

Let $\bar{B}$ be the band of idempotents of S. We know that the idempotents of W_B form a band B^* isomorphic to B; indeed the mapping $\xi | B : e \to (\rho_e, \lambda_e)$ is an isomorphism of B onto B^*. Denoting the inverse of $\xi | B$ by $\kappa : B^* \to B$, we define a mapping $\zeta : \bar{B} \to B$ by

$$(x, t)\zeta = x\kappa \quad ((x, t) \in \bar{B}).$$

It is clear that ζ is a homomorphism. To see that it is onto, note that for any e in B the element $(\rho_e, \lambda_e)\gamma_1{}^\natural$ is an idempotent in W_B/γ_1, and so there is a unique idempotent g in T such that $g\Psi = (\rho_e, \lambda_e)\gamma_1{}^\natural$. Then $((\rho_e, \lambda_e)g) \in \bar{B}$ and has image e under ζ.

To show that ζ is one–one, suppose that the elements (x, t), (y, u) in $\bar{B}$ are such that $(x, t)\zeta = (y, u)\zeta$. Then $x\kappa = y\kappa$ and so $x = y$ since κ is an isomorphism. Hence $x\gamma_1{}^\natural = y\gamma_1{}^\natural$ and so $t\Psi = u\Psi$ by the definition formula (3) of S. But t and u are idempotents of T and so, since Ψ is idempotent-separating, $t = u$. Thus $(x, t) = (y, u)$, and we conclude that ζ is an isomorphism of $\bar{B}$ onto B.

It is easy to see that $\pi : (x, t) \to t$ is a homomorphism of S into the inverse semigroup T. In fact, π maps onto T, since $\gamma_1{}^\natural$ maps W_B onto W_B/γ_1 and so for every t in T there is an element x in W_B such that $x\gamma_1{}^\natural = t\Psi$, that is, such that $(x, t) \in S$. If γ is the minimum inverse semigroup congruence on S, it follows that $\gamma \subseteq \pi \circ \pi^{-1}$ and that there is a homomorphism α of S/γ onto T such that

$$\begin{array}{ccccc} W_B & \xleftarrow{\xi} & S & \xrightarrow{\gamma^\natural} & S/\gamma & & \\ & & \pi\downarrow & \swarrow^{\alpha} & \downarrow\theta & & \\ & & S & \xrightarrow[\Psi]{} & W_B/\gamma_1 & \xrightarrow{\gamma_1{}^\natural} & W_B \end{array} \qquad (4)$$

commutes.

We know that the set of inverses of (x, t) in S is $V(x) \times \{t^{-1}\}$, where $V(x)$ is the set of inverses of the element x in W_B. Hence, using the characterization of γ by Proposition 2.2, we have that

$$\gamma = \{((x, t), (y, u)) \in S \times S : V(x) \times \{t^{-1}\} = V(y) \times \{u^{-1}\}\}$$
$$= \{((x, t), (y, u)) \in S \times S : t = u \text{ and } V(x) = V(y)\}.$$

On the other hand, $t = u$ implies $x\Psi = u\Psi$, which in turn implies $x\gamma_1 = y\gamma_1$ since (x, t), $(y, u) \in S$. Thus, using the characterization of γ_1 by Proposition 2.2, we have that if $t = u$ then it follows that $V(x) = V(y)$. Therefore,

$$\gamma = \{((x, t), (y, u)) \in S \times S : t = u\}$$
$$= \pi \circ \pi^{-1}$$

and so the mapping $\alpha : S/\gamma \to T$ in the diagram (4) is an isomorphism.

This completes the proof of Theorem.

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Myongji University